

# RETRACTIONS OF FREE MV-ALGEBRAS AND UNITAL $\ell$ -GROUPS

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**ABSTRACT.** A number of papers deal with the problem of counting the number of retractions of a structure  $S$  onto a substructure  $T$ . In the particular case when  $S$  is a free algebra, this number is  $\geq 1$  iff  $T$  is projective. In this paper we consider the case when  $T$  is a projective lattice-ordered abelian group with a distinguished strong order unit, or equivalently, a projective MV-algebra. Let  $A$  be a retract of the free  $n$ -generator MV-algebra  $\mathcal{M}([0, 1]^n)$  of McNaughton functions on  $[0, 1]^n$ . We prove that the number  $r(A)$  of retractions of  $\mathcal{M}([0, 1]^n)$  onto  $A$  is finite if, and only if, the maximal spectral space  $\mu_A$  is homeomorphic to a (Kuratowski) closed domain  $M$  of  $[0, 1]^n$ , in the sense that  $M = \text{cl}(\text{int}(M))$ . Further, the closed domain condition is decidable and  $r(A)$  is computable, once a retraction onto  $A$  is explicitly given. Thus every finitely generated projective MV-algebra  $B$  comes equipped with a new invariant  $\iota(B) = \sup\{r(A) \mid A \cong B \text{ for } A \text{ a retract of } \mathcal{M}([0, 1]^k)\}$ , where  $k$  is the smallest number of generators of  $B$ . We compute  $\iota(B)$  for many projective MV-algebras  $B$  considered in the literature. Various problems concerning retractions of free MV-algebras are shown to be decidable. Via the  $\Gamma$  functor, our results and computations automatically transfer to finitely generated projective abelian  $\ell$ -groups with a distinguished strong unit.

## 1. FOREWORD

Several papers deal with the problem of counting the number  $r(T)$  of *retractions* (= idempotent endomorphisms) of a structure  $S$  onto a substructure  $T \subseteq S$ . See, e.g., [5, 20, 26, 29], [16, p.174], [4, p.122]. In the particular case when  $S$  is a free algebra,  $r(T) \geq 1$  iff  $T$  is projective.

In this paper we will compute  $r(T)$  when  $T$  is a projective MV-algebra or equivalently, a projective *unital  $\ell$ -group*, which is short for “lattice-ordered abelian group with a distinguished strong order unit”. As a particular case of the equivalence  $\Gamma$  established in [22, Theorem 3.9], finitely presented MV-algebras are categorically equivalent to finitely presented unital  $\ell$ -groups. Further, both categories are dually equivalent to *rational polyhedra*, i.e., finite unions of simplexes with rational vertices in the same euclidean space  $\mathbb{R}^n$ ,  $n = 1, 2, \dots$ , with morphisms given by  $\mathbb{Z}$ -maps, i.e. piecewise-linear maps  $f$  with a finite number of linear pieces, such that each linear piece of  $f$  has integer coefficients, [21], [24, §3]. The synergy between these three categories has received increasing attention in the last few years, [7]–[11], [14], [18], [21], [24].

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Differently from finitely presented  $\ell$ -groups, finitely presented *unital*  $\ell$ -groups, as well as finitely presented MV-algebras  $A$  and their dual rational polyhedra, are endowed with a wealth of computable invariants, such as: the number of rational points of a given denominator  $d = 1, 2, \dots$  in the maximal spectral space  $\mu_A$ , [24, Proposition 3.15 and Theorem 4.16]; the rational measure of the  $t$ -dimensional part of  $\mu_A$ , ( $t = 0, \dots, \dim(\mu_A)$ ), [24, §14], [25]. None of these invariants makes sense for finitely presented  $\ell$ -groups.

A new numerical invariant, the *index*  $\iota(A)$ , will be introduced in this paper, by counting the maximum number of retractions of a free  $n$ -generator algebra onto  $A$ , where  $n$  is the smallest number of generators of  $A$ .

Not surprisingly, the isomorphism problem for finitely presented unital  $\ell$ -groups is still open, although Markov's celebrated unrecognizability theorem [27] is to the effect that the isomorphism problem for finitely presented  $\ell$ -groups is undecidable, [19].

Another main point of distinction between  $\ell$ -groups and unital  $\ell$ -groups is the characterization of finitely generated projectives. On the one hand, from the Baker-Beynon duality [1, 2, 3] one easily obtains that finitely generated projective  $\ell$ -groups coincide with the finitely presented ones. On the other hand, finitely generated projective unital  $\ell$ -groups (resp., finitely generated projective MV-algebras) are a proper subclass of finitely presented unital  $\ell$ -groups (resp., finitely presented MV-algebras). Their characterization is a tour de force in algebraic topology, [7, 8].

In this paper we focus on  $n$ -generator projective MV-algebras,  $n = 1, 2, \dots$ , using their rich algebraic, geometric, arithmetic and algorithmic structure. It is well known that any such MV-algebra is isomorphic to a retract  $A$  of the free MV-algebra  $\mathcal{M}([0, 1]^n)$  of McNaughton functions over the unit  $n$ -cube  $[0, 1]^n$ . Let  $r(A)$  denote the number of retractions of  $\mathcal{M}([0, 1]^n)$  onto  $A$ . In Theorem 7.4 we prove that  $r(A)$  is Turing computable.

Following Kuratowski, [15, p.20], we say that a subset  $D$  of a topological space  $X$  is a *closed domain* in  $X$  if  $D$  coincides with the closure of its interior in  $X$ , in symbols,  $\text{cl}(\text{int}(D)) = D$ . For any finitely generated projective MV-algebra  $B$ , letting  $k_B$  be the smallest number of its generators, we define the *index*  $\iota(B)$  as the sup of all  $r(A)$  as  $A$  ranges over retracts of  $\mathcal{M}([0, 1]^{k_B})$  isomorphic to  $B$ . Then in Corollary 4.3 we prove that  $\iota(B)$  is finite iff the maximal ideal space of  $B$  is homeomorphic to a closed domain in  $\mathbb{R}^{k_B}$ . Depending on  $B$ ,  $\iota(B)$  can be an arbitrarily large finite number, already in the two-dimensional case, (Theorem 5.1). Various estimates and computations of the multiplicity and of the index are carried on (respectively in §§3-6 and §7), and various related problems are shown to be Turing decidable.

Via the mentioned  $\Gamma$  equivalence, the results of this paper automatically transfer to finitely generated projective unital  $\ell$ -groups. Anyway, in this paper we will mostly work in the MV-algebraic framework, because all the algebraic machinery concerning finite presentations and projectives, (resp., all the algorithmic machinery needed to compute invariants) naturally arises from MV-algebras (resp., from the underlying Łukasiewicz calculus of MV-algebras). For all necessary background on MV-algebras we refer to the monographs [12] and [24].

## 2. POLYHEDRA AND RETRACTS OF FREE MV-ALGEBRAS AND UNITAL $\ell$ -GROUPS

A *rational polyhedron*  $P \subseteq \mathbb{R}^n$  is the union of finitely many simplexes in  $\mathbb{R}^n$  with rational vertices. By a  $\mathbb{Z}$ -map  $\zeta: P \rightarrow [0, 1]^m$  we mean a piecewise linear map where each linear piece has integer coefficients, and the number of linear pieces is finite. (Throughout this paper the adjective “linear” is understood in the affine sense.) A  $\mathbb{Z}$ -homeomorphism  $\theta$  of a rational polyhedron  $P \subseteq [0, 1]^n$  onto a rational

polyhedron  $Q \subseteq [0, 1]^m$  is a  $\mathbb{Z}$ -map of  $P$  onto  $Q$  such that also the inverse  $\theta^{-1}$  is a  $\mathbb{Z}$ -map. A  $\mathbb{Z}$ -map  $\sigma: [0, 1]^n \rightarrow [0, 1]^n$  is said to be a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  if it satisfies the idempotence condition  $\sigma \circ \sigma = \sigma$ . The set

$$R_\sigma = \text{range}(\sigma) \subseteq [0, 1]^n$$

is said to be a  $\mathbb{Z}$ -retract of  $[0, 1]^n$ .  $R_\sigma$  is a rational polyhedron, and we have the identity  $R_\sigma = \{x \in [0, 1]^n \mid x = \sigma(x)\}$ .

For  $n = 1, 2, \dots$ , we let  $\mathcal{M}([0, 1]^n)$  denote the MV-algebra of  $[0, 1]$ -valued  $\mathbb{Z}$ -maps defined over  $[0, 1]^n$ , equipped with the pointwise operations of the standard MV-algebra  $[0, 1]$ .  $\mathcal{M}([0, 1]^n)$  is a free MV-algebra, which throughout this paper comes equipped with the free generating set  $\{\pi_1, \dots, \pi_n\}$ , where  $\pi_i: [0, 1]^n \rightarrow [0, 1]$  is the  $i$ th coordinate map. Elements of  $\mathcal{M}([0, 1]^n)$  are known as *McNaughton functions*.

For any MV-term  $q(X_1, \dots, X_n)$  we write  $\hat{q}: [0, 1]^n \rightarrow [0, 1]$  for the McNaughton function associated to  $q$ . In the notation of [12, §3.1],  $\hat{q}$  is written  $q^{\mathcal{M}([0, 1]^n)}$ . In particular,  $\hat{X}_i$  is the  $i$ th coordinate function  $\pi_i: [0, 1]^n \rightarrow [0, 1]$ . More generally, for any  $n$ -tuple  $t = (t_1, \dots, t_n)$  of MV-terms, where all  $t_i$  are in the same variables  $X_1, \dots, X_n$ , we let  $\hat{t}$  denote the  $\mathbb{Z}$ -map  $(\hat{t}_1, \dots, \hat{t}_n): [0, 1]^n \rightarrow [0, 1]^n$ .

Following [21], let  $\mathcal{M}$  denote the functor from the category of rational polyhedra with  $\mathbb{Z}$ -maps to finitely presented MV-algebras, [24, §3], [21]. For any rational polyhedron  $P \subseteq [0, 1]^n$ , the MV-algebra  $\mathcal{M}(P)$  is defined by restricting to  $P$  every element of  $\mathcal{M}([0, 1]^n)$ , in symbols,  $\mathcal{M}(P) = \{f \upharpoonright P \mid f \in \mathcal{M}([0, 1]^n)\}$ , where  $\upharpoonright$  denotes restriction. Further, the action of  $\mathcal{M}$  on any  $\mathbb{Z}$ -map  $\sigma$  is given by

$$\mathcal{M}_\sigma = - \circ \sigma. \quad (1)$$

If in particular  $\sigma: [0, 1]^n \rightarrow [0, 1]^n$  is a  $\mathbb{Z}$ -retraction,  $\mathcal{M}_\sigma$  is a retraction that maps  $\mathcal{M}([0, 1]^n)$  onto the MV-subalgebra  $\text{gen}(\sigma_1, \dots, \sigma_n)$  of  $\mathcal{M}([0, 1]^n)$  generated by  $\sigma_1, \dots, \sigma_n$ . Thus by (1),  $\mathcal{M}_\sigma$  is the uniquely determined homomorphism of  $\mathcal{M}([0, 1]^n)$  into  $\mathcal{M}([0, 1]^n)$  extending the map  $\pi_i \mapsto \sigma_i$ , ( $i = 1, \dots, n$ ).

Conversely, for any retraction  $\epsilon: \mathcal{M}([0, 1]^n) \rightarrow \mathcal{M}([0, 1]^n)$ , the  $n$ -tuple  $\mathcal{Z}_\epsilon = (\epsilon(\pi_1), \dots, \epsilon(\pi_n)): [0, 1]^n \rightarrow [0, 1]^n$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$ . The range  $R_{\mathcal{Z}_\epsilon}$  of  $\mathcal{Z}_\epsilon$  is a rational polyhedron and coincides with the set  $\{x \in [0, 1]^n \mid x = \mathcal{Z}_\epsilon(x)\}$ .

It is easy to see that two maps  $\mathcal{M}$  and  $\mathcal{Z}$  are inverses of each other,

$$\mathcal{M}_{\mathcal{Z}_\epsilon} = \epsilon \quad \text{and} \quad \mathcal{Z}_{\mathcal{M}_\sigma} = \sigma. \quad (2)$$

Throughout we let  $\text{id}_X$  denote the identity map on a set  $X$ . By a *retract* we mean the range of a retraction.

**Theorem 2.1.** *Let  $\sigma = (\sigma_1, \dots, \sigma_n)$  be a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  onto the rational polyhedron  $R_\sigma$ . Let  $\text{gen}(\sigma_1, \dots, \sigma_n)$  be the retract of  $\mathcal{M}([0, 1]^n)$  associated with  $\sigma$ .*

- (a) *The map  $\tau \mapsto R_\tau$  yields a one-one correspondence between:*
  - $\mathbb{Z}$ -retractions  $\tau = (\tau_1, \dots, \tau_n)$  of  $[0, 1]^n$  such that  $\text{gen}(\tau_1, \dots, \tau_n) = \text{gen}(\sigma_1, \dots, \sigma_n)$ , and
  - rational polyhedra  $Q \subseteq [0, 1]^n$  such that  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} R_\sigma$ .
- (b) *Thus there exists a one-one correspondence between:*
  - retractions of  $\mathcal{M}([0, 1]^n)$  onto the MV-algebra  $\text{gen}(\sigma_1, \dots, \sigma_n)$ .
  - rational polyhedra  $Q \subseteq [0, 1]^n$  such that  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} R_\sigma$ , and

*Proof.* (a) Let  $\tau: [0, 1]^n \rightarrow [0, 1]^n$  be a  $\mathbb{Z}$ -retraction satisfying the condition

$$\text{gen}(\tau_1, \dots, \tau_n) = \text{gen}(\sigma_1, \dots, \sigma_n).$$

Then there are MV-terms  $t_1, \dots, t_n$  and  $s_1, \dots, s_n$  such that  $\tau_i = t_i(\sigma_1, \dots, \sigma_n)$  and  $\sigma_i = s_i(\tau_1, \dots, \tau_n)$ . Hence  $\hat{t} = (\hat{t}_1, \dots, \hat{t}_n)$  and  $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n): [0, 1]^n \rightarrow [0, 1]^n$  are  $\mathbb{Z}$ -maps satisfying

$$\sigma = \hat{s} \circ \tau \quad \text{and} \quad \tau = \hat{t} \circ \sigma. \quad (3)$$

*Claim.*  $\sigma \upharpoonright R_\tau$  is a  $\mathbb{Z}$ -homeomorphism onto  $R_\sigma$  satisfying the identity

$$(\sigma \upharpoonright R_\tau)^{-1} = \tau \upharpoonright R_\sigma. \quad (4)$$

As a matter of fact, let us pick an arbitrary  $x \in R_\sigma$ . The identities  $(\sigma \circ \tau)(x) = (\hat{s} \circ \tau \circ \tau)(x) = (\hat{s} \circ \tau)(x) = \sigma(x) = x$  show that  $\sigma \upharpoonright R_\tau$  is onto  $R_\sigma$ . Similarly, for all  $y \in R_\tau$  we have  $(\tau \circ \sigma)(y) = y$ . It follows that  $\sigma \upharpoonright R_\tau$  is one-one. The identity (4) is now immediate, and the claim is proved.

To complete the proof of (a), let us assume that, conversely,  $Q \subseteq [0, 1]^n$  is a rational polyhedron such that  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} R_\sigma$ . Let us write  $\zeta = (\zeta_1, \dots, \zeta_n)$  as an abbreviation of the  $\mathbb{Z}$ -homeomorphism  $(\sigma \upharpoonright Q)^{-1}$  of  $R_\sigma$  onto  $Q$ ,

$$\zeta = (\sigma \upharpoonright Q)^{-1}: R_\sigma \cong_{\mathbb{Z}} Q.$$

Observe that  $\zeta$  is piecewise linear with integer coefficients, and is defined over the rational polyhedron  $R_\sigma$ . For short,  $\zeta$  is a  $\mathbb{Z}$ -map on  $R_\sigma \subseteq [0, 1]^n$ . So by [24, Proposition 3.2] we have a  $\mathbb{Z}$ -map  $\bar{\zeta}: [0, 1]^n \rightarrow [0, 1]^n$  extending  $\zeta$ . By McNaughton theorem, [12, Theorem 9.1.5], for each  $i+1, \dots, n$ ,  $\bar{\zeta}_i$  is the McNaughton function of some  $n$ -variable MV-term. The composite map  $\rho = \zeta \circ \sigma$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  onto  $Q$ , because  $\zeta \circ \sigma \circ \zeta \circ \sigma = \zeta \circ \text{id}_{R_\sigma} \circ \sigma = \zeta \circ \sigma$ . Since  $R_\sigma$  is a  $\mathbb{Z}$ -retract of  $[0, 1]^n$  then so is the rational polyhedron  $Q = \zeta(R_\sigma)$ . From  $\rho = \zeta \circ \sigma$  we get  $\text{gen}(\rho_1, \dots, \rho_n) \subseteq \text{gen}(\sigma_1, \dots, \sigma_n)$ . From  $\sigma = \zeta^{-1} \circ \rho = \sigma \circ \rho$  we get  $\text{gen}(\sigma_1, \dots, \sigma_n) \subseteq \text{gen}(\rho_1, \dots, \rho_n)$ . Further, by (3) and (4) we can write  $\zeta \circ \sigma = \tau \upharpoonright R_\sigma \circ \sigma = \tau \circ \sigma = \hat{t} \circ \sigma \circ \sigma = \hat{t} \circ \sigma = \tau$ , and  $R_{\zeta \circ \sigma} = \zeta \circ \sigma([0, 1]_n) = \zeta(R_\sigma) = Q$ . Thus the maps  $\tau \mapsto R_\tau$  and  $Q \mapsto \zeta \circ \sigma$  are inverse of each other, and (a) is proved.

(b) This immediately follows from (a) and (2).  $\square$

For the proof of Theorem 2.3 below, we record the following elementary fact:

**Lemma 2.2.** *Let  $\eta: [0, 1]^n \rightarrow [0, 1]^n$  be a  $\mathbb{Z}$ -map and  $P, Q \subseteq [0, 1]^n$  be rational polyhedra satisfying the following conditions:*

- (i) *both  $\text{int}(P)$  and  $\text{int}(Q)$  are connected;*
- (ii)  *$P = \text{cl}(\text{int}(P))$  and  $Q = \text{cl}(\text{int}(Q))$ ;*
- (iii)  *$\eta(P) = \eta(Q)$ ;*
- (iv)  *$\eta \upharpoonright P: P \cong_{\mathbb{Z}} \eta(P)$  and  $\eta \upharpoonright Q: Q \cong_{\mathbb{Z}} \eta(Q)$ .*

*Then either  $P = Q$  or  $\text{int}(P) \cap \text{int}(Q) = \emptyset$ .*

*Proof.* By way of contradiction, let us assume  $P \neq Q$  and there is  $x \in \text{int}(P) \cap \text{int}(Q)$ . Without loss of generality assume that  $y \in P \setminus Q$  for some  $y$ . By (ii),  $P = \text{cl}(\text{int}(P))$ , whence we may insist that  $y \in \text{int}(P)$ . Since by (i) the interior of  $P$  is an open connected subset of  $\mathbb{R}^n$ , it is also path connected. (See Figure 1.) Let  $\gamma: [0, 1] \rightarrow P$  be a path such that  $\gamma([0, 1]) \subseteq \text{int}(P)$ ,  $\gamma(0) = x$ , and  $\gamma(1) = y$ . Since  $\gamma$  is continuous and  $Q$  is closed, the set  $J = \{\delta \in [0, 1] \mid \gamma(\delta) \in Q\} \subseteq [0, 1]$  is closed. Let  $\lambda$  be the largest element of  $J$ . From  $\gamma(1) = y \notin Q$  we get  $\lambda < 1$ . Let  $z = \gamma(\lambda)$ . Then  $z \in \text{int}(P)$  and  $z \in Q \setminus \text{int}(Q)$ . By (iv),  $\eta$  maps  $z$  to a point  $\eta(z)$  that simultaneously belongs to the interior of  $\eta(P)$  and to the boundary of  $\eta(Q)$ , which contradicts (iii).  $\square$

Up to isomorphism, any  $n$ -generator projective MV-algebra  $B$  has the form  $\mathcal{M}(P) = \{f \upharpoonright P \mid f \in \mathcal{M}([0, 1]^n)\}$  for some  $\mathbb{Z}$ -retract  $P$  of  $[0, 1]^n$ . Specifically, by [8, Theorem 5.1] or [24, Proposition 17.5], there is a  $\mathbb{Z}$ -retraction  $\sigma$  of  $[0, 1]^n$  such that  $B \cong \mathcal{M}(R_\sigma) \cong \text{range}(\mathcal{M}_\sigma) = \text{range}(- \circ \sigma)$ . By [24, Corollary 4.18], the  $\mathbb{Z}$ -retract  $Q = R_\sigma$  is homeomorphic to the maximal spectral space  $\mu(B)$ . If another  $\mathbb{Z}$ -retract  $Q'$  of  $[0, 1]^n$  is chosen such that  $B \cong \mathcal{M}(Q')$ , then  $Q$  is  $\mathbb{Z}$ -homeomorphic to  $Q'$  ([24, Corollary 3.10]). Thus in particular  $Q$  is a closed domain in  $[0, 1]^n$  iff so

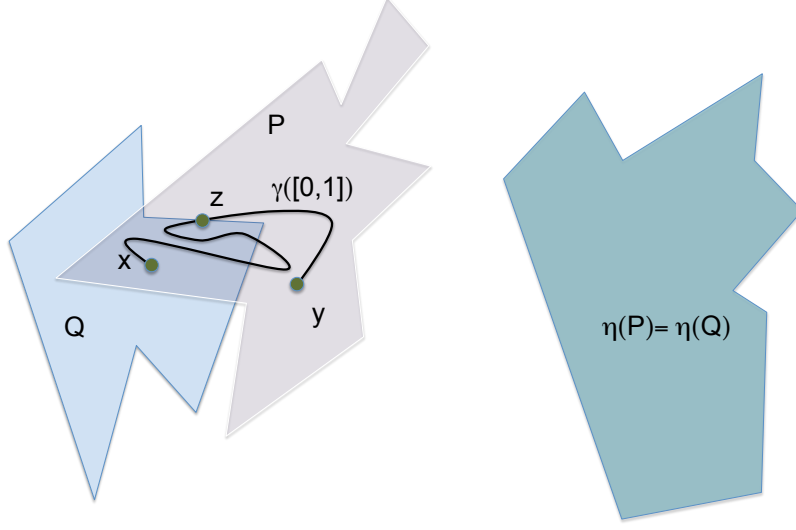


FIGURE 1. The path  $\gamma: [0, 1] \rightarrow \text{int}(P)$  in the proof of Lemma 2.2 joins  $x \in \text{int}(P) \cap \text{int}(Q)$  and  $y \in P \setminus Q$ , and has a nonempty intersection with the boundary of  $Q$ .

is  $Q'$ . (See [15, p.20] for this terminology, going back to Kuratowski.) This state of affairs can be unambiguously described by saying that the maximal spectral space  $\mu_B$  is a closed domain in  $[0, 1]^n$ .

**Theorem 2.3.** *Suppose  $A$  is a retract of  $\mathcal{M}([0, 1]^n)$  and  $\mu_A$  is a closed domain in  $[0, 1]^n$ . Then the number of retractions of  $\mathcal{M}([0, 1]^n)$  onto  $A$  is finite.*

*Proof.* Let us choose a retraction  $\epsilon$  of  $\mathcal{M}([0, 1]^n)$  onto  $A$ , along with its associated  $\mathbb{Z}$ -retraction  $\mathcal{Z}_\epsilon = \sigma$  as given by (2). Since  $R_\sigma$  is a polyhedron (it is compact and) the connected components of  $\text{int}(R_\sigma) \subseteq [0, 1]^n$  are finitely many. Let  $O_{\sigma,1}, \dots, O_{\sigma,k} \subseteq \text{int}(R_\sigma) \subseteq [0, 1]^n$  be the list of these connected components.

With reference to the notation (1) for the functor  $\mathcal{M}$ , let  $\zeta$  be a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  such that  $\mathcal{M}_\zeta$  is a retraction of  $\mathcal{M}([0, 1]^n)$  onto  $A$ . By Theorem 2.1,  $\sigma \upharpoonright R_\zeta$  is  $\mathbb{Z}$ -homeomorphism onto  $R_\sigma$ . Therefore,  $\text{int}(R_\zeta)$  has  $k$  connected components  $O_{\zeta,1}, \dots, O_{\zeta,k}$ , and we can write  $\sigma(O_{\tau,j}) = O_{\sigma,j}$ . We have  $\mathbb{Z}$ -homeomorphisms

$$\sigma \upharpoonright \text{cl}(O_{\zeta,j}): \text{cl}(O_{\zeta,j}) \rightarrow \text{cl}(O_{\sigma,j}), \quad (j = 1, \dots, k). \quad (5)$$

Let the family  $\mathcal{O}$  of open sets in  $[0, 1]^n$  be defined by

$$\mathcal{O} = \{O_{\zeta,j} \mid j = 1, \dots, k, \text{ and the map } - \circ \zeta \text{ is a retraction of } \mathcal{M}([0, 1]^n) \text{ onto } A\}.$$

Let  $O_{\zeta,j}, O_{\zeta',j'} \in \mathcal{O}$ . If  $O_{\zeta,j} \neq O_{\zeta',j'}$ , then either  $j \neq j'$  or  $\zeta \neq \zeta'$ . If  $j \neq j'$ , then  $\sigma(O_{\zeta,j}) \cap \sigma(O_{\zeta',j'}) = \emptyset$ , whence  $O_{\zeta,j} \cap O_{\zeta',j'} = \emptyset$ . If  $j = j'$ , then  $\zeta \neq \zeta'$ . From Lemma 2.2, (with  $\eta = \sigma$ ,  $P = \text{cl}(O_{\zeta,j})$ , and  $Q = \text{cl}(O_{\zeta',j'})$ ) it follows that  $O_{\zeta,j} \cap O_{\zeta',j'} = \emptyset$ . Therefore, the elements of  $\mathcal{O}$  are pairwise disjoint.

Since  $\mathbb{Z}$ -homeomorphisms preserve the Lebesgue measure of  $n$ -dimensional polyhedra in  $[0, 1]^n$  ([24, Lemma 14.3], [23, Theorem 2.1(iii)]), by (5) each  $O_{\zeta,j} \in \mathcal{O}$  has the same ( $n$ -dimensional) Lebesgue measure as  $O_{\sigma,j}$ , because  $O_{\sigma,j}$  has the same Lebesgue measure as  $\text{cl}(O_{\zeta,j})$ . Let  $O_{\sigma,j}$  be chosen among  $O_{\sigma,1}, \dots, O_{\sigma,k}$  as having the smallest  $n$ -dimensional Lebesgue measure. Say that  $\lambda$  is its measure. Since the

elements of  $\mathcal{O}$  are pairwise disjoint, we have

$$\text{number of elements in } \mathcal{O} \leq \lfloor 1/\lambda \rfloor = \max\{l \in \mathbb{Z} \mid l \leq 1/\lambda\}. \quad (6)$$

By Theorem 2.1, the number  $r(A)$  of retractions of  $[0, 1]^n$  onto  $A$  satisfies the inequality

$$r(A) \leq \binom{\lfloor 1/\lambda \rfloor}{k}. \quad (7)$$

This completes the proof.  $\square$

Throughout we let  $\mathcal{M}_{\mathbb{R}}([0, 1]^n)$  denote the unital  $\ell$ -group of piecewise linear functions  $f: [0, 1]^n \rightarrow \mathbb{R}$ , where each linear piece of  $f$  has integer coefficients. In view of the categorical equivalence  $\Gamma$  between unital  $\ell$ -groups and MV-algebras, [22, Theorem 3.9], such notions as “free unital  $\ell$ -group” and “finitely presented  $\ell$ -group” make perfect sense, not only as the  $\Gamma$ -correspondents of free and finitely presented MV-algebras, but also from the categorical viewpoint, (respectively see [22, Corollary 4.16] and [11, Remark 5.10].)

The maximal spectral space  $\mu_G$  of every unital  $\ell$ -group  $(G, u)$  is canonically homeomorphic to the maximal spectral space of its associated MV-algebra  $\Gamma(G, u)$ , [12, §7.2]. Precisely as in the case of MV-algebras, it makes perfect mathematical sense to say that  $\mu_G$  is a closed domain in  $[0, 1]^n$ .

By [22, Theorem 4.15],  $\Gamma(\mathcal{M}_{\mathbb{R}}([0, 1]^n)) = \mathcal{M}([0, 1]^n)$ . Thus by [12, §7.2], up to unital  $\ell$ -isomorphism every finitely generated projective unital  $\ell$ -group has the form  $\mathcal{M}_{\mathbb{R}}(P) = \{f \upharpoonright P \mid f \in \mathcal{M}_{\mathbb{R}}([0, 1]^n)\}$  for some  $n = 1, 2, \dots$  and  $\mathbb{Z}$ -retract  $P$  of  $[0, 1]^n$ .

**Corollary 2.4.** *Given a retract  $(G, u)$  of  $\mathcal{M}_{\mathbb{R}}([0, 1]^n)$ , suppose  $\mu_G$  is a closed domain in  $[0, 1]^n$ . Then the number of retractions of  $\mathcal{M}_{\mathbb{R}}([0, 1]^n)$  onto  $(G, u)$  is finite.*

*Proof.* Immediate from Theorem 2.3, using the preservation properties of the  $\Gamma$  equivalence, [12, §7.2].  $\square$

### 3. THE INDEX OF A PROJECTIVE MV-ALGEBRA AND OF A UNITAL $\ell$ -GROUP

**Definition 3.1.** The *multiplicity*  $r(A)$  of a retract  $A$  of  $\mathcal{M}([0, 1]^n)$  is the number of distinct retractions of  $\mathcal{M}([0, 1]^n)$  onto  $A$  if this number is finite, and  $\infty$  otherwise. The *index*  $\iota(B) \in \{1, 2, \dots\} \cup \{\infty\}$  of a finitely generated projective MV-algebra  $B$  is the supremum of the multiplicities of all retracts  $A \cong B$  of  $\mathcal{M}([0, 1]^k)$ , with  $k$  the smallest number of generators of  $B$ . One similarly defines the index of finitely generated projective unital  $\ell$ -groups.

**Proposition 3.2.** (a) *Let  $P \subseteq [0, 1]^n$  be a  $\mathbb{Z}$ -retract and a closed domain in  $[0, 1]^n$ . Let  $m$  be the maximum number of  $\mathbb{Z}$ -homeomorphic pairwise disjoint copies of  $P$  in  $[0, 1]^n$ . Then  $\iota(\mathcal{M}(P)) \geq m$ .*

(b) *An upper bound for the index  $\iota(\mathcal{M}(P))$  is given by (7).*

*Proof.* (a) Our assumption  $P = \text{cl}(\text{int}(P))$  ensures that  $n$  is the smallest number of generators of  $\mathcal{M}(P)$ . As a matter of fact, if  $\mathcal{M}(P)$  had  $n - 1$  generators (absurdum hypothesis) then by [12, Theorem 3.6.7]  $\mathcal{M}(P)$  would be isomorphic to an MV-algebra of the form  $\mathcal{M}(X)$  for some closed subset  $X$  of  $[0, 1]^{n-1}$ . By [24, Corollary 4.18] the maximal spectral space  $\mu_{\mathcal{M}(X)}$  is homeomorphic to  $X$ , whence its dimension is  $\leq n - 1$ . On the other hand, from the isomorphism  $\mathcal{M}(P) \cong \mathcal{M}(X)$  we get the homeomorphism  $P \cong X$ , so  $\dim(P) \leq n - 1$ , thus contradicting the assumption that  $P$  is a closed domain in  $[0, 1]^n$ .

Let  $Q_1, Q_2, \dots, Q_m$  be a (maximal) set of pairwise disjoint  $\mathbb{Z}$ -homeomorphic copies of  $P$  in  $[0, 1]^n$ . Since by [24, Corollary 3.10]  $\mathcal{M}(Q_1) \cong \mathcal{M}(P)$  and the index is an isomorphism invariant, we may assume  $Q_1 = P$  without loss of generality. If

$m = 1$  we have nothing to prove. So assume  $m \geq 2$ . For each  $i = 2, \dots, m$  there is a  $\mathbb{Z}$ -homeomorphism  $\eta_i$  of  $Q_i$  onto  $Q_1$ . For completeness let us set  $\eta_1 = \text{id}_P$ . Since the  $Q_j$  are pairwise disjoint ( $j = 1, \dots, m$ ) the set  $\bigcup_{j=1}^m \eta_j$  is a  $\mathbb{Z}$ -map of  $\bigcup_{j=1}^m Q_j$  onto  $P$ . By [24, Proposition 3.2(ii)] there is a  $\mathbb{Z}$ -map  $\eta: [0, 1]^n \rightarrow [0, 1]^n$  simultaneously extending each  $\eta_j$ . Pick a  $\mathbb{Z}$ -retraction  $\sigma$  of  $[0, 1]^n$  onto  $P$ . Then the composite map  $\sigma \circ \eta$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  onto  $P$ , and for each  $j = 1, \dots, m$  the restriction  $\sigma \circ \eta|_{Q_j} = \sigma \circ \eta_j = \eta_j$  is a  $\mathbb{Z}$ -homeomorphism of  $Q_j$  onto  $P$ . By Theorem 2.1, the multiplicity of the retract  $A = \text{gen}(\sigma_1, \dots, \sigma_n)$  is  $\geq m$ . By [24, Lemma 3.6]  $A \cong \mathcal{M}(P)$ , whence the desired conclusion follows by definition of the index, recalling that  $n$  is the smallest number of generators of  $\mathcal{M}(P)$ .

(b) By [24, Lemma 14.3] or [23, Theorem 2.1(iii)],  $\mathbb{Z}$ -homeomorphisms preserve the Lebesgue measure of  $n$ -dimensional polyhedra in  $[0, 1]^n$ . By [24, Corollary 3.10],  $\mathcal{M}(P) \cong \mathcal{M}(Q)$  implies  $P \cong_{\mathbb{Z}} Q$ .  $\square$

**Corollary 3.3.** *The index of every finitely generated free MV-algebra is 1. Similarly, for every  $n = 1, 2, \dots$ , the index of the free unital  $\ell$ -group  $\mathcal{M}_{\mathbb{R}}([0, 1]^n)$  is 1. In particular, the two-element MV-algebra  $\{0, 1\}$  is the free MV-algebra over 0 generators. Its index is equal to 1.*

The following example explains why in the definition of the index of  $B$  we restrict to those isomorphic copies of  $B$  which are retracts of  $\mathcal{M}([0, 1]^k)$  with  $k$  the smallest number of generators of  $B$ . As in [24, p.11], or [25], for any rational point  $r \in \mathbb{R}^n$ , the denominator  $\text{den}(r)$  is defined by

$$\text{den}(r) = \text{least common denominator of the coordinates of } r. \quad (8)$$

**Example 3.4.** For  $n \geq 1$  let  $\text{cyl}(n, \mathcal{M}([0, 1])) \subseteq \mathcal{M}([0, 1]^n)$  be the isomorphic copy of  $\mathcal{M}([0, 1])$  obtained by cylindrifying each  $f \in \mathcal{M}([0, 1])$  into the function  $c \in \mathcal{M}([0, 1]^n)$  given by  $c(x, x_2, \dots, x_n) = f(x)$  for all  $(x, x_2, \dots, x_n) \in [0, 1]^n$ . By Corollary 3.3 the index of the free MV-algebra  $\mathcal{M}([0, 1])$  is 1. We claim that the multiplicity of its isomorphic copy  $\text{cyl}(2, \mathcal{M}([0, 1]))$  is  $\infty$ . Let the  $\mathbb{Z}$ -retraction  $\xi = (\xi_1, \xi_2): [0, 1]^2 \rightarrow [0, 1]^2$  be given by  $\xi_1(x, y) = x$ ,  $\xi_2(x, y) = 0$ .  $\xi$  projects any point of the unit square onto the  $x$ -axis. A direct inspection shows that  $\xi$  preserves the denominator of a rational point of  $(x, y) \in [0, 1]^2$  iff the denominator of  $y$  is a divisor of the denominator of  $x$ . This is the case, in particular, when the point  $(x, y)$  belongs to the graph  $W$  of a McNaughton function  $f$  in  $\mathcal{M}([0, 1])$ , because every linear piece of  $f$  has integer coefficients. By [24, Proposition 3.15],  $\xi$  acts  $\mathbb{Z}$ -homeomorphically over the broken line  $W \subseteq [0, 1]^2$ . There are countably many such broken lines  $W$ , one for each  $f \in \mathcal{M}([0, 1])$ . By Theorem 2.1(c) there are countably many retractions of  $\mathcal{M}([0, 1]^2)$  onto  $\text{cyl}(2, \mathcal{M}([0, 1]))$ . Thus the multiplicity of  $\text{cyl}(2, \mathcal{M}([0, 1]))$  is  $\infty$ , and our claim is proved. One similarly proves that the multiplicity of  $\text{cyl}(n, \mathcal{M}([0, 1]))$  is  $\infty$  for each  $n \geq 2$ . As already noted in Corollary 3.3,  $\iota(\mathcal{M}([0, 1])) = 1$  whence  $\iota(\text{cyl}(n, \mathcal{M}([0, 1]))) = 1$  for each  $n$ .

The proof of the following result is immediate:

**Proposition 3.5.** *Let  $\sigma = (\sigma_1, \dots, \sigma_n)$  be a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$ , and  $\alpha$  a  $\mathbb{Z}$ -homeomorphism of  $[0, 1]^n$  onto  $[0, 1]^n$ . Then the range of the composite map  $\alpha \circ \sigma$  is a  $\mathbb{Z}$ -retract, and so is its  $\mathbb{Z}$ -homeomorphic copy  $R_\sigma \subseteq [0, 1]^n$ . If  $R_\sigma$  is  $n$ -dimensional,  $n$  is the smallest number of generators of the retract  $\text{gen}(\sigma_1, \dots, \sigma_n)$  of  $\mathcal{M}([0, 1]^n)$ . Letting  $\tau = \alpha \circ \sigma \circ \alpha^{-1}$ , it follows that  $\tau$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$ , and the two isomorphic retracts  $\text{gen}(\sigma_1, \dots, \sigma_n)$  and  $\text{gen}(\tau_1, \dots, \tau_n)$  have equal multiplicities and equal indexes.*

Figure 2 shows the special case of Proposition 3.5 for  $n = 2$ , where  $\alpha$  is Panti's  $\mathbb{Z}$ -homeomorphism, [13] and  $\sigma = \pi_1 \wedge \neg \pi_1: [0, 1]^2 \rightarrow [0, 1]^2$ .

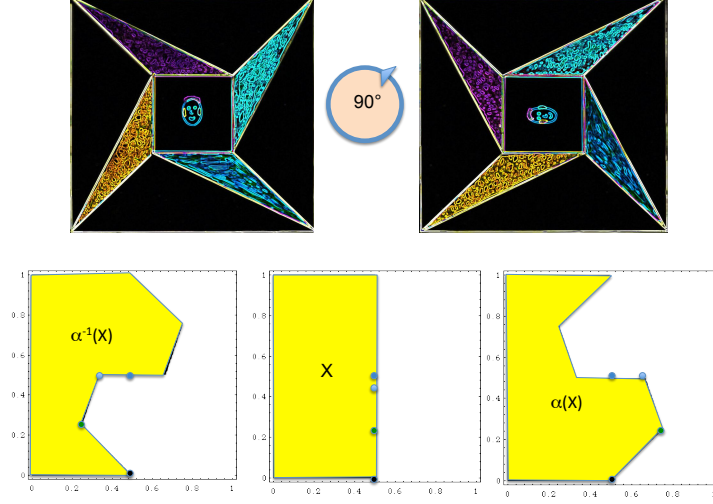


FIGURE 2. Panti's  $\mathbb{Z}$ -homeomorphism  $\alpha: [0, 1]^2 \rightarrow [0, 1]^2$  and the image  $\alpha(X)$  of the rectangle  $X = \{(x, y) \in [0, 1]^2 \mid x \leq 1/2\}$ . As explained in [13],  $\alpha$  rotates counterclockwise by  $90^\circ$  the inner square with vertices  $(1/3, 1/3), (1/3, 2/3), (2/3, 1/3), (2/3, 2/3)$ ; the rubber edges of the eight triangles in the picture are modified accordingly. The perimeter of the square is kept fixed by  $\alpha$ , and so is the central point  $(1/2, 1/2)$ . The  $\mathbb{Z}$ -retraction  $\sigma: (x, y) \mapsto x \wedge \neg x$  sends  $[0, 1]^2$  onto  $X$ . The map  $\alpha \circ \sigma$  sends  $[0, 1]^2$  onto  $\alpha(X)$ , but is not a  $\mathbb{Z}$ -retraction. The  $\mathbb{Z}$ -retraction  $\tau = \alpha \circ \sigma \circ \alpha^{-1}$  sends  $[0, 1]^2$  onto  $\alpha(X)$ . So both  $X$  and  $\alpha(X)$  are  $\mathbb{Z}$ -retracts of  $[0, 1]^2$ .

The effective computability of the index of a one-generator projective MV-algebra is taken care of by the following easy result:

**Proposition 3.6.** *Let  $B \not\cong \{0, 1\}$  be a one-generator projective MV-algebra.*

- (a) *For a unique rational  $0 < r \in [0, 1]$  we have the isomorphism  $B \cong \mathcal{M}([0, r])$ . Then  $\iota(B) \in \{1, 2\}$ . Further,  $\iota(B) = 2$  iff  $r \leq 1/2$ .*
- (b) *In equivalent algebraic-topological terms,  $\iota(B) = 2$ , unless the maximal spectral space  $\mu_B$  contains an element  $\mathfrak{m}$  such that  $B/\mathfrak{m} \cong \{0, 1/2, 1\}$  and  $\mu_B \setminus \{\mathfrak{m}\}$  is disconnected—in which case  $\iota(B) = 1$ .*

*Proof.* (a) The first statement is a particular case of [24, Proposition 17.5], upon noting that every  $\mathbb{Z}$ -retract of  $[0, 1]$  is  $\mathbb{Z}$ -homeomorphic to  $\mathcal{M}([0, r])$  for some  $r \in \mathbb{Q} \cap [0, 1]$ . Further,  $r > 0$ , for otherwise  $B$  would be isomorphic to the two-element MV-algebra. In case  $r > 1/2$  the measure-theoretic argument in the proof of Theorem 2.3 shows that  $\iota(A) = 1$ . On the other hand, if  $r \leq 1/2$ , the only other rational polyhedron in  $[0, 1]$  which is  $\mathbb{Z}$ -homeomorphic to  $[0, r]$  is  $[1 - r, 1]$ . By Theorem 2.1(b) and Proposition 3.2,  $\iota(\mathcal{M}([0, r])) = 2$ .

(b) This is just a reformulation of part (a) in the light of the spectral theory of MV-algebras, [24, §4.5], and the duality between finitely presented MV-algebras and rational polyhedra, [24, §3], [21].  $\square$

While the index is invariant under isomorphisms, in the following example we present two isomorphic retracts of  $\mathcal{M}([0, 1])$  having different multiplicities.



**Example 3.7.** Let the  $\mathbb{Z}$ -retraction of  $[0, 1]$  onto  $[0, 1/2]$  be given by  $\sigma(x) = x \wedge \neg x$ . The retract  $A = \text{gen}(\sigma) = \text{gen}(\pi_1 \wedge \neg \pi_1) \subseteq \mathcal{M}([0, 1])$  is the MV-algebra of all one-variable McNaughton functions  $f$  such that  $f(1-x) = f(x)$ . Since the restriction of  $\sigma$  to  $[1/2, 1]$  is a  $\mathbb{Z}$ -homeomorphism onto  $[0, 1/2]$  and  $(\sigma \upharpoonright [1/2, 1])^{-1} = \pi_1 \vee \neg \pi_1$ , by Theorem 2.1 the map  $\mathcal{M}_{\pi_1 \vee \neg \pi_1}$  is a second retraction  $\mathcal{M}([0, 1])$  onto  $A$ . Moreover,  $\mathcal{M}_\sigma$  and  $\mathcal{M}_\rho$  are the only two retractions of  $\mathcal{M}([0, 1])$  onto  $A$ . Thus  $r(A) = 2$ . Let  $\tau: [0, 1] \rightarrow [0, 1]$  be given by  $\tau(x) = (x \wedge \neg x) \wedge ((\neg x \oplus \neg x) \odot (\neg x \oplus \neg x))$ . Then  $\tau$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]$  onto  $[0, 1/2]$ . Let  $B = \text{gen}(\tau)$ . We have  $A \cong B \cong \mathcal{M}([0, 1/2])$ . For no other segment  $J$  other than  $[0, 1/2]$  it is the case that  $\tau \upharpoonright J$  is a  $\mathbb{Z}$ -homeomorphism of  $J$  onto  $[0, 1/2]$ . By Theorem 2.1,  $r(B) = 1$ .

#### 4. WHEN THE MAXIMAL SPECTRAL SPACE IS NOT A CLOSED DOMAIN IN $[0, 1]^n$

For any rational  $m$ -simplex  $T = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ , let us display each vertex  $v_j$  of  $T$  as  $(a_{j1}/b_{j1}, \dots, a_{jn}/b_{jn})$ , for uniquely determined integers  $a_{jt}, b_{jt}$  ( $t = 1, \dots, n$ ) such that  $b_{jt} > 0$ . With the notation of (8) we let the *homogeneous correspondent*  $\tilde{v}_j$  of  $v_j$  be defined by

$$\tilde{v}_j = \text{den}(v_j)(a_{j1}/b_{j1}, \dots, a_{jn}/b_{jn}, 1) \in \mathbb{Z}^{n+1}.$$

Conversely,  $v_j$  is said to be the *affine correspondent* of  $\tilde{v}_j$ .

An  $m$ -simplex  $U = \text{conv}(w_0, \dots, w_m) \subseteq \mathbb{R}^n$  is said to be *regular* if it is rational and the set of integer vectors  $\{\tilde{w}_0, \dots, \tilde{w}_m\}$  can be extended to a basis of the free abelian group  $\mathbb{Z}^{n+1}$ .

For every simplicial complex  $\Sigma$  the point-set union of the simplexes of  $\Sigma$  is called the *support* of  $\Sigma$ , and is denoted  $|\Sigma|$ . We also say that  $\Sigma$  is a *triangulation* of  $|\Sigma|$ . A simplicial complex is said to be a *regular triangulation* (of its support) if all its simplexes are regular. Regular triangulations (called “unimodular” in [23]) are the affine counterparts of the regular, or nonsingular, fans of toric algebraic geometry, [17]. Suppose  $\Sigma$  and  $\Theta$  are two simplicial complexes with the same support, and every simplex of  $\Theta$  is contained in some simplex of  $\Sigma$ . Then  $\Theta$  is said to be a *subdivision* of  $\Sigma$ .

Let  $\Sigma$  be a simplicial complex and  $b \in |\Sigma| \subseteq \mathbb{R}^n$ . Following [17, III, 2.1], the simplicial complex  $\Sigma_{(b)}$  is obtained by the following procedure: replace every simplex  $S \in \Sigma$  containing  $b$  by the set of all simplexes of the form  $\text{conv}(b, F)$ , where  $F$  is any face of  $S$  that does not contain  $b$ . The subdivision  $\Sigma_{(b)}$  of  $\Sigma$  is known as the *blow-up*  $\Sigma_{(b)}$  of  $\Sigma$  at  $b$ .

For any  $m \geq 1$  and regular  $m$ -simplex  $U = \text{conv}(w_0, \dots, w_m) \subseteq \mathbb{R}^n$  the *Farey median* of  $U$  is the affine correspondent of the vector  $\tilde{w}_0 + \dots + \tilde{w}_m \in \mathbb{Z}^{n+1}$ . In the particular case when  $\Sigma$  is a regular triangulation and  $b$  is the *Farey median* of a simplex of  $\Sigma$ , the blow-up  $\Sigma_{(b)}$  is regular.

The short proof of the following proposition is a template for the main construction in the proof of Theorem 4.2, yielding a converse of Theorem 2.3.

**Proposition 4.1.** *There is a retract of  $\mathcal{M}([0, 1]^2)$  having an infinite index.*

*Proof.* Let  $L$  be the union of the two edges  $\text{conv}((0, 1), (0, 0))$  and  $\text{conv}((1, 0), (0, 0))$ . Let  $\rho = (\rho_1, \rho_2): [0, 1]^2 \rightarrow L$  be the  $\mathbb{Z}$ -retraction of  $[0, 1]^2$  onto  $L$  given by

$$\rho(x, y) = (x \ominus y, y \ominus x) = \begin{cases} (0, y - x) & \text{if } y \geq x; \\ (x - y, 0) & \text{if } x \geq y. \end{cases}$$

A direct inspection shows that  $\rho$  sends each point  $(x, y) \in [0, 1]^2$  to the point of  $L$  whose coordinates are  $x - \min(x, y)$  and  $y - \min(x, y)$ . Geometrically,  $\rho$  moves down

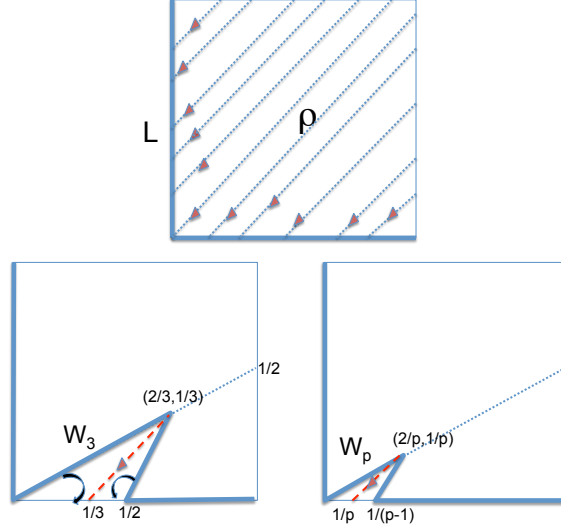


FIGURE 3. The  $\mathbb{Z}$ -retraction  $\rho$  in the proof of Proposition 4.1. Each broken line  $W_p$ ,  $p \geq 3$ , is mapped  $\mathbb{Z}$ -homeomorphically onto  $L$  by  $\rho$ . The index of the MV-algebra  $\mathcal{M}(L)$  is  $\infty$ .

by 45 degrees in the south-west direction each point  $(x, y) \in [0, 1]^2$ , by subtracting the same quantity  $\min(x, y)$  to each coordinate.

*Claim.* The MV-algebra  $A = \text{gen}(\rho_1, \rho_2) \subseteq \mathcal{M}([0, 1]^2)$  has infinite multiplicity.

As a matter of fact, (see Figure 3) for each integer  $p \geq 3$  let the broken line  $W_p \subseteq [0, 1]^2$  be the union of the segment  $\text{conv}((0, 1), (0, 0))$  with the three segments

$$\text{conv}\left((0, 0), \left(\frac{2}{p}, \frac{1}{p}\right)\right), \quad \text{conv}\left(\left(\frac{2}{p}, \frac{1}{p}\right), \left(\frac{1}{p-1}, 0\right)\right), \quad \text{conv}\left(\left(\frac{1}{p-1}, 0\right), (1, 0)\right).$$

It is not hard to check that  $\rho$  is a  $\mathbb{Z}$ -homeomorphism of  $W_p$  onto  $L$ . To this purpose one notes that the triangle  $\text{conv}((0, 0), (\frac{2}{p}, \frac{1}{p}), (\frac{1}{p-1}, 0))$  is the union of the regular triangles  $\text{conv}((0, 0), (\frac{2}{p}, \frac{1}{p}), (\frac{1}{p}, 0))$   $\text{conv}((\frac{1}{p-1}, 0), (\frac{2}{p}, \frac{1}{p}), (\frac{1}{p}, 0))$ . Further

- $\rho$  fixes the segment  $\text{conv}((0, 1), (0, 0))$ ;
- $\rho$  maps  $\text{conv}((0, 0), (\frac{2}{p}, \frac{1}{p}))$  one-one onto  $\text{conv}((0, 0), (\frac{1}{p}, 0))$ ;
- $\rho$  maps  $\text{conv}((\frac{2}{p}, \frac{1}{p}), (\frac{1}{p-1}, 0))$  one one onto  $\text{conv}((\frac{1}{p}, 0), (\frac{1}{p-1}, 0))$ ;
- $\rho$  fixes  $\text{conv}((\frac{1}{p-1}, 0), (1, 0))$ .

By [24, Lemma 3.7, Proposition 3.15],  $\rho$  is an invertible  $\mathbb{Z}$ -map of  $W_p$  onto  $L$ . To see that the multiplicity of  $A$  is infinite, recall Theorem 2.1, and let  $p$  range over all integers  $\geq 3$ . Having thus settled our claim, the proof of the proposition is complete.  $\square$

Following [8, Definition 4.1], a triangulation  $\Delta$  of a rational polyhedron  $P$  is said to be *strongly regular* if it is regular and for each maximal simplex  $T$  of  $\Delta$  the greatest common divisor of the denominators of the vertices of  $T$  is equal to 1.  $P$  is called *strongly regular* if it has a strongly regular triangulation. Then every regular triangulation of  $P$  is strongly regular ([8, Remark 5.1]). Every  $\mathbb{Z}$ -retract of  $[0, 1]^n$  is strongly regular, [8, Theorem 5.2(iii)].

**Theorem 4.2.** *Fix a  $\mathbb{Z}$ -retraction  $\rho = (\rho_1, \dots, \rho_n)$  of  $[0, 1]^n$ . Let  $P = R_\rho$  be the range of  $\rho$ . If some (equivalently, every) triangulation of  $P$  contains a maximal  $m$ -simplex with  $m < n$  then the number of retractions of  $\mathcal{M}([0, 1]^n)$  onto  $\text{gen}(\rho_1, \dots, \rho_n)$  is infinite.*

*Proof.* Since  $P$  is a  $\mathbb{Z}$ -retract of  $[0, 1]^n$  then  $\mathcal{M}(P)$  projective, [8, Theorem 5.1], [24, Proposition 17.5(ii)]. By Theorem 2.1 there is a one-one correspondence between the set of retractions of  $\mathcal{M}([0, 1]^n)$  onto the MV-algebra  $\text{gen}(\rho_1, \dots, \rho_n) \subseteq \mathcal{M}([0, 1]^n)$  and the set of rational polyhedra  $R \subseteq [0, 1]^n$  such that the restriction  $\rho|_R$  is a  $\mathbb{Z}$ -homeomorphism of  $R$  onto  $P$ . Let us say that any such  $R$  is a  *$\mathbb{Z}$ -homeomorphism domain* for  $\rho$ . So it suffices to show that the number of such domains  $R$  is infinite.

Let  $\Delta$  be a regular triangulation of  $P$ . The existence of  $\Delta$  follows from [24, Corollary 2.10]. By assumption,  $\Delta$  has a maximal  $m$ -simplex  $T$  with  $m < n$ . It follows that  $m \geq 1$ , for otherwise by [8, Theorem 5.2(i)-(ii)] the  $\mathbb{Z}$ -retract  $P$  would coincide with a vertex of  $[0, 1]^n$ , so  $\mathcal{M}(P)$  is the free MV-algebra over 0 generators, and  $n = m = 0$ , which is impossible.

Since  $P$  is a  $\mathbb{Z}$ -retract of  $[0, 1]^n$ ,  $P$  is strongly regular, [8, Theorem 5.2(iii)]. Thus, for some prime number  $p$  there exists a rational point  $c$  of denominator  $p$  with

$$c \in \text{relint } T. \quad (9)$$

Actually, such  $c$  exists for all sufficiently large primes  $p$ , because  $T$  is a strongly regular  $m$ -simplex with  $m > 0$ .

As another consequence of the strong regularity of  $P$ , the affine hull  $\text{aff}(T) \subseteq \mathbb{R}^n$  of  $T$  contains some integer point of  $\mathbb{Z} \subseteq \mathbb{R}^n$ , [6, Theorem 4.17]. Then the construction of [10, Lemma 5] yields integer points  $j_0, \dots, j_m \in \mathbb{Z}^n$  such that  $\text{aff}(T) = \text{aff}(\text{conv}(j_0, \dots, j_m))$  and the  $m$ -simplex  $I = \text{conv}(j_0, \dots, j_m) \subseteq \mathbb{R}^n$  is regular and contains  $c$  in its relative interior.

Let  $\mathcal{G}_n = \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$  denote the  $n$ -dimensional affine group over the integers. By [10, Lemma 1] some function  $\gamma \in \mathcal{G}_n$  maps  $\text{aff}(T)$  one-one onto the  $m$ -dimensional space

$$F_m = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{m+1} = \dots = x_n = 0\}.$$

Thus the  $m$ -simplex  $\gamma(I)$  lies in  $F_m$ , and we can write without loss of generality

$$\gamma(I) = \text{conv}(0, \underbrace{(1, 0, \dots, 0)}_{n-1 \text{ zeros}}, \underbrace{(0, 1, 0, \dots, 0)}_{n-2 \text{ zeros}}, \dots, \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{n-m \text{ zeros}}).$$

Since  $\gamma$  is a (linear)  $\mathbb{Z}$ -homeomorphism, it preserves denominators of rational points and maps regular simplexes one-one onto regular simplexes, [24, Proposition 3.15]. Let us display the point  $c' = \gamma(c)$  as follows:

$$c' = (c'_1, \dots, c'_m, \underbrace{0, \dots, 0}_{n-m \text{ zeros}}) = (a_1/p, \dots, a_m/p, 0, \dots, 0)$$

for suitable relatively prime integers  $0 \leq a_1, \dots, a_m \leq p$ . Note that  $\text{den}(c') = \text{den}(c) = p$ . By (9),

$$c' \in \text{relint}(\gamma(T)). \quad (10)$$

We next define the point  $l \in \mathbb{R}^n$  by

$$l = (c'_1, \dots, c'_m, 1/p, \underbrace{0, \dots, 0}_{n-m-1 \text{ zeros}}) = (a_1/p, \dots, a_m/p, 1/p, 0, \dots, 0). \quad (11)$$

Permuting, if necessary, the coordinates in  $\mathbb{R}^n$ , for all sufficiently large primes  $p$  we can safely assume

$$\gamma^{-1}(l) \in [0, 1]^n. \quad (12)$$

Since  $P$  is a polyhedron and  $T \in \Delta$ , then by (9) for all sufficiently small  $\epsilon > 0$  the closed ball  $B_{\epsilon,c}$  of radius  $\epsilon$  centered at  $c$  satisfies the condition

$$B_{\epsilon,c} \cap P \subseteq T. \quad (13)$$

The affine transformation  $\gamma$  sends  $B_{\epsilon,c}$  one-one onto an  $n$ -dimensional ellipsoid  $\gamma(B_{\epsilon,c})$  containing the point  $\gamma(c)$  in its relative interior. Further, by (13) we can write

$$\gamma(B_{\epsilon,c}) \cap \gamma(P) \subseteq \gamma(T). \quad (14)$$

The map  $\rho' = \gamma \circ \rho \circ \gamma^{-1}$  is a  $\mathbb{Z}$ -retraction of the  $n$ -parallelepiped  $\gamma([0,1]^n)$  onto the rational polyhedron  $\gamma(P)$ . By (14), all points sufficiently close to  $c'$  are mapped by  $\rho'$  into points lying in the  $m$ -simplex  $\gamma(T)$ . For all sufficiently small  $\epsilon > 0$  the piecewise linear map  $\rho'$  is linear over  $\gamma(B_{\epsilon,c})$ . A continuity argument recalling (10) ensures that the point  $l^* = \rho'(l)$  lies in the relative interior of  $\gamma(T)$ , because  $l$  tends to  $c'$  as  $p$  tends to  $\infty$ .

The De Concini-Procesi theorem in the version of [24, Theorem 5.3] (or the affine version of the desingularization procedure of [17, p.70]) yields a regular triangulation  $\nabla$  of  $\gamma(T)$  such that  $l^*$  is a vertex of some simplex of  $\nabla$ . The set  $S$  of  $m$ -simplexes of  $\nabla$  is now defined by

$$S = \{B \mid B \in \nabla \text{ is an } m\text{-simplex having } l^* \text{ among its vertices}\}.$$

Fix now  $B \in S$  and write  $B = \text{conv}(v_0, v_1, \dots, v_m)$  for suitable points  $v_i \in \mathbb{R}^n$ . For each  $i = 0, \dots, m$  let us display the homogeneous correspondent  $\tilde{v}_i \in \mathbb{Z}^{n+1}$  of vertex  $v_i$  as follows:

$$\tilde{v}_i = (b_{i1}, \dots, b_{im}, \underbrace{0, \dots, 0}_{n-m \text{ zeros}}, d_i).$$

From the regularity of  $B \in S \subseteq \nabla$  we get

$$\det \begin{pmatrix} b_{01} & b_{02} & \dots & b_{0m} & d_0 \\ b_{11} & b_{12} & \dots & b_{1m} & d_1 \\ \dots & \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mm} & d_m \end{pmatrix} = \pm 1. \quad (15)$$

Recalling (11) we can similarly write

$$\tilde{l} = (a_1, \dots, a_m, 1, \underbrace{0, \dots, 0}_{n-m-1 \text{ zeros}}, p).$$

Let  $\mathbf{b}_1, \dots, \mathbf{b}_m, \mathbf{d}$  be the column vectors of the integer matrix (15). We then have

$$\det \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_m & 0 & \mathbf{d} \\ a_1 & a_2 & \dots & a_m & 1 & p \end{pmatrix} = \pm 1,$$

showing the regularity of the  $(m+1)$ -simplex  $P_B = \text{conv}(B, l)$  for every  $B \in S$ . The basis of the pyramid  $P_B$  is the  $m$ -simplex  $B \subseteq F_m$ . The lateral  $m$ -surface of  $P_B$  is the point set union of all  $m$ -simplexes of  $P_B$  having  $l$  as a vertex. Let the  $(m+1)$ -dimensional pyramid  $P_S \subseteq \mathbb{R}^n$  be defined by

$$P_S = \bigcup_{B \in S} P_B.$$

Its basis  $B_S$  is the point set union of the bases  $B$ 's, for all  $B \in S$ . Let  $L_B^* \subseteq L_B$  be obtained by stripping  $L_B$  of all  $m$ -simplexes of  $P_B$  having  $l^*$  as a vertex. Then the lateral surface  $L_S$  of  $P_S$  is given by

$$L_S = \text{cl} \bigcup_{B \in S} L_B^* = \bigcup_{B \in S} \text{cl}(L_B^*).$$

Since the  $\mathbb{Z}$ -retraction  $\rho'$  is linear over the ellipsoid  $\gamma(B_{\epsilon,c})$  then  $\rho'$  maps  $L_S$  one-one onto  $B_S$ . Intuitively,  $\rho'$  collapses the lateral surface of  $P_S$  one-one onto its basis  $B_S$ .

This can be directly verified for each  $B \in S$ , noting that  $\rho'$  maps  $\text{cl}(L_B^*)$  one-one onto  $B$ . Since  $\rho'$  preserves the denominators of the vertices of each  $m$ -simplex  $P_B$ , and  $P_B$  is regular, then by [24, Proposition 3.15]  $\rho'$  maps  $\mathbb{Z}$ -homeomorphically  $\text{cl}(L_B^*)$  onto  $B$ . Thus  $\rho'$  maps  $\mathbb{Z}$ -homeomorphically  $L_S$  onto  $B_S$ . Further,  $\rho'$  is identity over  $\gamma(P) \setminus B_S$ , whence  $\rho'$  is a  $\mathbb{Z}$ -homeomorphism of  $(\gamma(P) \setminus B_S) \cup L_S$  onto  $\gamma(P)$ .

In conclusion, the set  $(\gamma(P) \setminus B_S) \cup L_S$  is a  $\mathbb{Z}$ -homeomorphism domain of  $\rho'$ . Going back via  $\gamma^{-1}$  we see that the  $\mathbb{Z}$ -retraction  $\rho$  sends  $(P \setminus \gamma^{-1}(B_S)) \cup \gamma^{-1}(L_S)$   $\mathbb{Z}$ -homeomorphically onto  $P$ . (Note that (12) ensures that  $\gamma^{-1}(L_S)$  lies in the  $n$ -cube). The choice of  $c \in \text{relint}(T)$  and of the large prime  $p$  being arbitrary, it follows that there are infinitely many  $\mathbb{Z}$ -homeomorphism domains of  $\rho$ . By Theorem 2.1 the number of retractions of  $\mathcal{M}([0, 1]^n)$  onto  $\text{gen}(\rho_1, \dots, \rho_n)$  is infinite.  $\square$

Combining the foregoing theorem with Theorem 2.3 we immediately obtain:

**Corollary 4.3.** *Let  $k$  be the smallest number of generators of a finitely generated projective MV-algebra  $B$ . Then the index of  $B$  is finite iff the maximal spectral space of  $B$  is homeomorphic to a regular domain in  $[0, 1]^k$ .*

*Proof.* Identify  $B$  with  $\mathcal{M}(P)$  for some  $\mathbb{Z}$ -retract  $P$  of  $[0, 1]^k$ .

( $\Rightarrow$ ) If the maximal spectral space of  $B$  is not homeomorphic to a regular domain in  $[0, 1]^k$ , then the same holds for its homeomorphic copy  $P$ . As a consequence, every (equivalently, some) triangulation  $\Delta$  of  $P$  contains some maximal  $l$ -simplex with  $l < k$ . By the foregoing theorem, the index of  $B$  is infinite.

( $\Leftarrow$ ) If the maximal spectral space of  $B$  is homeomorphic to a regular domain in  $[0, 1]^k$ , then so is its homeomorphic copy  $P$ . Now apply Theorem 2.3.  $\square$

## 5. ARBITRARILY HIGH FINITE INDEX

**Theorem 5.1.** *For every  $j = 1, 2, \dots$  there is retract  $A_j$  of  $\mathcal{M}([0, 1]^2)$  such that the maximal spectral space of  $A_j$  is a closed domain in  $[0, 1]^2$  and  $j < \iota(A_j) \in \mathbb{Z}$ .*

*Proof.* For each rational point  $x = (x_1, x_2) \in [0, 1]^2$  let  $d = \text{den}(x)$  be the least common multiple of the denominators of  $x_1$  and  $x_2$ . Then for uniquely determined integers  $n_1, n_2$  we can write  $x_1 = n_1/d$  and  $x_2 = n_2/d$ . Throughout this proof we will specify  $x$  in terms of its *homogeneous integer coordinates* as in [24, §2.1]. Identifying  $x$  with its homogeneous correspondent we will write

$$x = (n_1/d, n_2/d) = [n_1, n_2, d]. \quad (16)$$

We will also write  $o$  for the origin  $[0, 0, 1]$  in  $\mathbb{R}^2$ .

The proof amounts to a construction of  $\mathbb{Z}$ -retracts  $\sigma^{(1)}, \sigma^{(2)}, \dots$  of  $[0, 1]^2$  such that the multiplicity of the retract  $A_n = \text{range}(- \circ \sigma^{(n)})$  is  $> 2^n$ . We assume familiarity with regular triangles, regular triangulations, and Farey blow-ups [24, §2.2, §5.1].

*Step 0.*

Let the regular triangles  $U_1, V_1 \subseteq [0, 1]^2$  be defined by

$$V_1 = \text{conv}(o, [1, 1, 1], [0, 1, 1]) \text{ and } U_1 = \text{conv}(o, [1, 0, 1], [1, 1, 1]).$$

Let  $\zeta^{(1)}: V_1 \rightarrow U_1$  be the unique linear extension of the map

$$o \mapsto o, [1, 1, 1] \mapsto [1, 1, 1], [0, 1, 1] \mapsto [1, 0, 1].$$

By [24, Lemma 3.7, Corollary 3.10],  $\zeta^{(1)}$  is a  $\mathbb{Z}$ -homeomorphism of  $V_1$  onto  $U_1$ , in symbols,  $\zeta^{(1)}: V_1 \cong_{\mathbb{Z}} U_1$ . Next let  $\rho^{(1)} = \sigma^{(1)} = \zeta^{(1)} \cup \text{id}_{U_1}$ . Then  $\sigma^{(1)}$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]^2$  onto  $U_1$ , acting  $\mathbb{Z}$ -homeomorphically over  $V_1$ . By Theorem 2.1, the multiplicity of the retract  $A_1 = \text{range}(- \circ \sigma^{(1)})$  is equal to  $2^1$ .

The *Fibonacci sequence*  $1, 1, 2, 3, 5, 8, 13, \dots$  be defined by

$$F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}. \quad (17)$$

*Step 1.*

Let  $\Sigma_1$  be the regular simplicial complex given by  $U_1$  and all its faces. Let  $b_1$  be the Farey median of the edge of  $U_1$  opposite to the origin  $o$ . Then the blow-up of  $\Sigma_1$  at  $b_1$  yields a regular simplicial complex, whose maximal triangles  $V_2, U_2$  are given by

$$V_2 = \text{conv}(o, [2, 1, F_3], [1, 1, F_2]) \text{ and } U_2 = \text{conv}(o, [1, 0, F_1], [2, 1, F_3]).$$

Let  $\zeta^{(2)}: V_2 \rightarrow U_2$  be the unique linear extension of the map

$$o \mapsto o, [2, 1, F_3] \mapsto [2, 1, F_3], [1, 1, F_2] \mapsto [1, 0, F_1].$$

Then  $\zeta^{(2)}$  is a  $\mathbb{Z}$ -homeomorphism of  $V_2$  onto  $U_2$ , in symbols,  $\zeta^{(2)}: V_2 \cong_{\mathbb{Z}} U_2$ . Next let  $\rho^{(2)} = \zeta^{(2)} \cup \text{id}_{U_2}$ . This is a  $\mathbb{Z}$ -retraction of  $U_1$  onto  $U_2$  acting  $\mathbb{Z}$ -homeomorphically over  $V_2$ . Let

$$\sigma^{(2)} = \rho^{(2)} \circ \sigma^{(1)} = \rho^{(2)} \circ \rho^{(1)}.$$

This is a  $\mathbb{Z}$ -retraction of  $[0, 1]^2$  onto  $U_2$  acting  $\mathbb{Z}$ -homeomorphically over the following  $2^2$  triangles:  $U_2, V_2, (\zeta^{(1)})^{-1}(U_2), (\zeta^{(1)})^{-1}(V_2)$ . By Theorem 2.1, the multiplicity of the retract  $A_2$  of  $\mathcal{M}([0, 1]^2)$  defined by  $A_2 = \text{range}(- \circ \sigma^{(2)})$  is equal to  $2^2$ .

*Step 2.*

Let  $\Sigma_2$  be the regular simplicial complex given by the triangle

$$U_2 = \text{conv}(o, [1, 0, F_2], [2, 1, F_3])$$

and all its faces. In homogeneous coordinates, let  $b_2 = [3, 1, F_4]$  be the Farey median of the edge  $\text{conv}([1, 0, F_2], [2, 1, F_3])$  of  $U_2$  opposite to the origin  $o$ . Then the blow-up of  $\Sigma_2$  at  $b_2$  yields a regular simplicial complex, whose maximal triangles  $V_3, W_3$  are given by

$$V_3 = \text{conv}(o, [3, 1, F_4], [2, 1, F_3]) \text{ and } W_3 = \text{conv}(o, [1, 0, F_2], [3, 1, F_4]).$$

We now let  $[1, 0, F_3]$  be the Farey median of  $[1, 0, F_2]$  and  $o = [0, 0, F_1]$ . Let  $\mathcal{W}_3$  be the (regular) simplicial complex given by  $W_3$  and all its faces. By blowing-up  $\mathcal{W}_3$  at  $[1, 0, F_3]$  we obtain a regular triangulation of  $W_3$  whose maximal triangles  $U_3$  and  $T_3$  are given by

$$U_3 = \text{conv}(o, [1, 0, F_3], [3, 1, F_4]) \text{ and } T_3 = \text{conv}([1, 0, F_3], [1, 0, F_2], [3, 1, F_4]).$$

By construction  $U_2 = V_3 \cup W_3 = V_3 \cup U_3 \cup T_3$ .

Let  $\zeta^{(3)}: V_3 \rightarrow U_3$  be the unique linear extension of the map

$$o \mapsto o, [3, 1, F_4] \mapsto [3, 1, F_4], [2, 1, F_3] \mapsto [1, 0, F_3].$$

As above,  $\zeta^{(3)}$  is a  $\mathbb{Z}$ -homeomorphism of  $V_3$  onto  $U_3$ , in symbols,  $\zeta^{(3)}: V_3 \cong_{\mathbb{Z}} U_3$ .

Let  $\lambda^{(3)}: T_3 \rightarrow U_3$  be the unique linear extension of the map

$$[1, 0, F_3] \mapsto [1, 0, F_3], [3, 1, F_4] \mapsto [3, 1, F_4], [1, 0, F_2] \mapsto o.$$

Then the map  $\rho^{(3)} = \zeta^{(3)} \cup \lambda^{(3)} \cup \text{id}_{U_3}$  is a  $\mathbb{Z}$ -retraction of  $U_2$  onto  $U_3$  acting  $\mathbb{Z}$ -homeomorphically over  $V_3$  and, trivially, over  $U_3$ . (Actually,  $\rho^{(3)}$  also acts  $\mathbb{Z}$ -homeomorphically over  $T_3$ , but for our purposes it is sufficient to restrict attention to the action of  $\rho^{(3)}$  over the two triangles  $V_3$  and  $U_3$ .) The map

$$\sigma^{(3)} = \rho^{(3)} \circ \sigma^{(3)} = \rho^{(3)} \circ \rho^{(2)} \circ \rho^{(1)}$$

is a  $\mathbb{Z}$ -retraction of  $[0, 1]^2$  onto  $U_3$  acting  $\mathbb{Z}$ -homeomorphically over (among others) the following  $2^3$  triangles:

$$U_3, V_3, (\zeta^{(2)})^{-1}(U_3), (\zeta^{(2)})^{-1}(V_3), (\zeta^{(1)})^{-1}(\text{the foregoing 4 triangles}). \quad (18)$$

By Theorem 2.1(c), the multiplicity of the retract  $A_3$  of  $\mathcal{M}([0, 1]^2)$  defined by  $A_3 = \text{range}(- \circ \sigma^{(3)})$  is  $\geq 2^3$ .

*Step 3.*

Let the regular simplex  $\Sigma_3$  consist of the triangle

$$U_3 = \text{conv}(o, [1, 0, F_3], [3, 1, F_4])$$

and all its faces. In homogeneous coordinates, let  $b_3 = [4, 1, F_5]$  be the Farey median of the edge  $\text{conv}([1, 0, F_3], [3, 1, F_4])$  of  $U_3$  opposite to the origin  $o$ . Then the blow-up of  $\Sigma_3$  at  $b_3$  yields a regular simplicial complex, whose maximal triangles  $V_4, W_4$  are given by

$$V_4 = \text{conv}(o, [4, 1, F_5], [3, 1, F_4]) \text{ and } W_4 = \text{conv}(o, [1, 0, F_3], [4, 1, F_5]).$$

We now let  $[1, 0, F_4]$  be the Farey median of  $[1, 0, F_3]$  and  $o = [0, 0, F_1]$ .

Let  $\mathcal{W}_4$  be the (regular) simplicial complex given by  $W_4$  and all its faces. By blowing-up  $\mathcal{W}_4$  at  $[1, 0, F_4]$ , we obtain a regular triangulation of  $W_4$  whose maximal triangles  $U_4$  and  $T_4$  are given by

$$U_4 = \text{conv}(o, [1, 0, F_4], [4, 1, F_5]) \text{ and } T_4 = \text{conv}([1, 0, F_4], [1, 0, F_3], [4, 1, F_5]).$$

Observe that  $U_3 = V_4 \cup W_4 = V_4 \cup U_4 \cup T_4$ .

Let  $\zeta^{(4)}: V_4 \rightarrow U_4$  be the unique linear extension of the map

$$o \mapsto o, [4, 1, F_5] \mapsto [4, 1, F_5], [3, 1, F_4] \mapsto [1, 0, F_4].$$

As above,  $\zeta^{(4)}$  is a  $\mathbb{Z}$ -homeomorphism of  $V_4$  onto  $U_4$ , in symbols,  $\zeta^{(4)}: V_4 \cong_{\mathbb{Z}} U_4$ .

Let  $\lambda^{(4)}: T_4 \rightarrow U_4$  be the unique linear extension of the map

$$[1, 0, F_4] \mapsto [1, 0, F_4], [4, 1, F_5] \mapsto [4, 1, F_5], [1, 0, F_3] \mapsto o.$$

Then the map

$$\rho^{(4)} = \zeta^{(4)} \cup \lambda^{(4)} \cup \text{id}_{U_4}$$

is a  $\mathbb{Z}$ -retraction of  $U_3$  onto  $U_4$  acting  $\mathbb{Z}$ -homeomorphically over  $V_4$ . The map

$$\sigma^{(4)} = \rho^{(4)} \circ \sigma^{(4)} = \rho^{(4)} \circ \rho^{(3)} \circ \rho^{(2)} \circ \rho^{(1)}$$

is a  $\mathbb{Z}$ -retraction of  $[0, 1]^2$  onto  $U_4$  acting  $\mathbb{Z}$ -homeomorphically over the following  $2^4$  triangles:

$$U_4, V_4, (\zeta^{(3)})^{-1}(U_4), (\zeta^{(3)})^{-1}(V_4), \text{ etc. etc. etc. unfolding.}$$

(As in the previous step,  $\sigma^{(4)}$  acts  $\mathbb{Z}$ -homeomorphically over other triangles, but for our present purposes it is convenient to restrict attention to these  $2^4$  only. See Figure 4.) By Theorem 2.1(c), the multiplicity of the retract  $A_4$  of  $\mathcal{M}([0, 1]^2)$  defined by  $A_4 = \text{range}(- \circ \sigma^{(4)})$  is  $\geq 2^4$ .

*Step  $n - 1$ , ( $n = 5, 6, \dots$ ).*

Inductively let the regular simplex  $\Sigma_{n-1}$  consist of the triangle

$$U_{n-1} = \text{conv}(o, [1, 0, F_{n-1}], [n-1, 1, F_n])$$

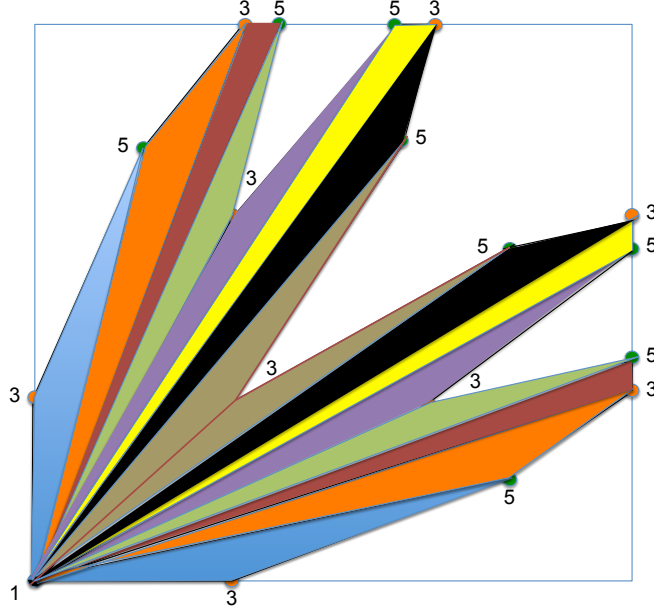


FIGURE 4. Sixteen  $\mathbb{Z}$ -homeomorphism domains of the map  $\sigma^{(4)}$  of Step 3 in the proof of Theorem 5.1. Each  $\mathbb{Z}$ -homeomorphism domain is a regular triangle whose vertices have denominators 1,3,5.

and all its faces. In homogeneous coordinates, let  $b_{n-1} = [n, 1, F_{n+1}]$  be the Farey median of the edge  $\text{conv}([1, 0, F_{n-1}], [n-1, 1, F_n])$  of  $U_{n-1}$  opposite to the origin  $o$ . Then the blow-up of  $\Sigma_{n-1}$  at  $b_{n-1}$  yields a regular simplicial complex, whose maximal simplexes  $V_n, W_n$  are given by

$$V_n = \text{conv}(o, [n, 1, F_{n+1}], [n-1, 1, F_n]) \text{ and } W_n = \text{conv}(o, [1, 0, F_{n-1}], [n, 1, F_{n+1}]).$$

Let the regular triangle  $U_n \subseteq W_n$  be given by  $U_n = \text{conv}(o, [1, 0, F_n], [n, 1, F_{n+1}])$ . Let  $\zeta^{(n)}: V_n \rightarrow U_n$  be the unique linear extension of the map

$$o \mapsto o, \quad [n, 1, F_{n+1}] \mapsto [n, 1, F_{n+1}], \quad [n-1, 1, F_n] \mapsto [1, 0, F_n].$$

The regularity of  $V_n$  and  $U_n$  ensures that  $\zeta^{(n)}$  is a  $\mathbb{Z}$ -homeomorphism of  $V_n$  onto  $U_n$ , in symbols,  $\zeta^{(n)}: V_n \cong_{\mathbb{Z}} U_n$ .

For each  $j = 0, \dots, F_{n-2} - 1$ , let the triangle  $T_{n,j}$  be defined by

$$T_{n,j} = \text{conv}([n, 1, F_{n+1}], [1, 0, F_{n-1} + j], [1, 0, F_{n-1} + j + 1]).$$

A direct verification shows that every  $T_{n,j}$  is regular. As a matter of fact, the triangle  $W_n = \text{conv}(o, [1, 0, F_{n-1}], [n, 1, F_{n+1}])$  is regular; the points  $[1, 0, F_{n-1} + 1], [1, 0, F_{n-1} + 2], \dots, [1, 0, F_{n-1} + F_{n-2} - 1], [1, 0, F_{n-1} + F_{n-2}] = [1, 0, F_n]$ , are obtained by taking the (always Farey) median  $[1, 0, F_{n-1} + 1]$  of  $o$  and  $[1, 0, F_{n-1}]$ , and then taking the median  $[1, 0, F_{n-1} + 2]$  of  $o$  and  $[1, 0, F_{n-1} + 1], \dots$ , and finally taking the median  $[1, 0, F_n]$  of  $o$  and  $[1, 0, F_{n-1} - 1] = [1, 0, F_{n-1} + F_{n-2} - 1]$ . Let  $\mathcal{W}_n$  be the regular simplicial complex given by  $W_n$  and all its faces. Then  $U_n$  and the  $T_{n,j}$  are the maximal simplexes of a regular triangulation of  $W_n$ , which is obtained from  $\mathcal{W}_n$  by consecutive Farey blow-ups as described in Figure 5. Observe that  $U_{n-1} = V_n \cup W_n = V_n \cup U_n \cup \bigcup_j T_{n,j}$ .



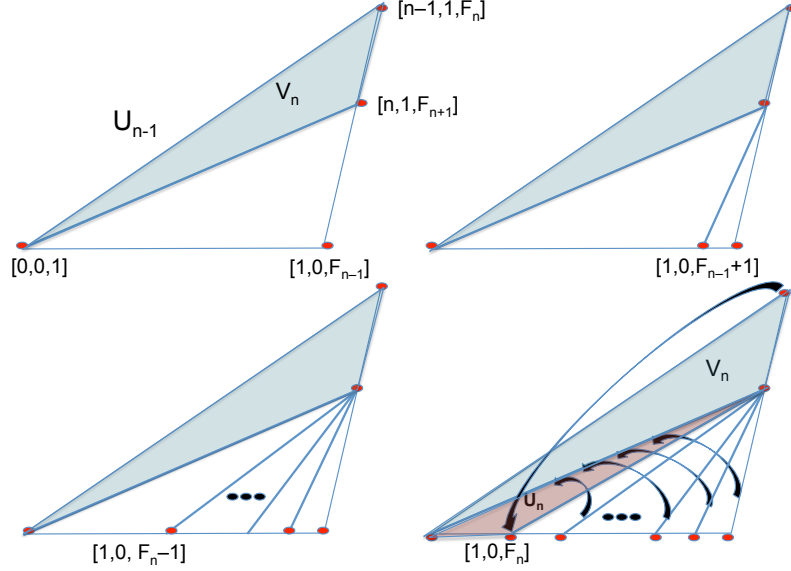


FIGURE 5. The sequence of Farey blow-ups and retractions in the proof of Theorem 5.1.

For each  $j = 0, \dots, F_{n-2} - 1$ , let

$$\lambda_j^{(n)}: \text{conv}([1, 0, F_n], [n, 1, F_{n+1}], [1, 0, F_{n-1} + j]) \rightarrow U_n$$

be the unique linear extension of the map

$$[1, 0, F_n] \mapsto [1, 0, F_n], \quad [n, 1, F_{n+1}] \mapsto [n, 1, F_{n+1}], \quad [1, 0, F_{n-1} + j] \mapsto o.$$

By [24, Lemma 3.7], each  $\lambda_j^{(n)}$  is linear with integer coefficients (i.e.,  $\lambda_j^{(n)}$  is a linear  $\mathbb{Z}$ -map) sending the regular triangle  $\text{conv}([n, 1, F_{n+1}], [1, 0, F_n], [1, 0, F_{n-1} - 1])$  onto  $U_n$ , and mapping all other triangles  $T_j$  onto the segment  $\text{conv}(o, [n, 1, F_{n+1}]) \subseteq U_n$ . The map

$$\rho^{(n)} = \zeta^{(n)} \cup \text{id}_{U_3} \cup \bigcup_{j=0}^{-1+F_{n-2}} \lambda_j^{(n)}$$

is a  $\mathbb{Z}$ -retraction of  $U_{n-1}$  onto  $U_n$  acting  $\mathbb{Z}$ -homeomorphically over  $V_n$ . The map

$$\sigma^{(n)} = \rho^{(n)} \circ \sigma^{(n-1)} = \rho^{(n)} \circ \rho^{(n-1)} \circ \dots \circ \rho^{(1)}$$

is a  $\mathbb{Z}$ -retraction of  $[0, 1]^2$  onto  $U_n$ . Generalizing (18),  $\sigma^{(n)}$  is a  $\mathbb{Z}$ -homeomorphism onto  $U_n$  of each one of the following  $2^n$  triangles:  $U_n, V_n, (\zeta^{(n-1)})^{-1}(U_n), (\zeta^{(n-1)})^{-1}(V_n), (\zeta^{(n-2)})^{-1}$ (the foregoing four triangles),  $\dots, (\zeta^{(2)})^{-1}$ (the foregoing  $2^{n-2}$  triangles)  $(\zeta^{(1)})^{-1}$  (the foregoing  $2^{n-1}$  triangles). Actually,  $\sigma^{(n)}$  is a  $\mathbb{Z}$ -homeomorphism also of other triangles onto  $U_n$ , but these are irrelevant to our purposes. By Theorem 2.1, the multiplicity of the retract  $A_n = \text{range}(- \circ \sigma^{(n)})$  of  $\mathcal{M}([0, 1]^2)$  is  $\geq 2^n$ . Since the area of  $U_n$  is  $> 0$ , by (6) the multiplicity of  $A_n$  is finite.

Iterating this inductive procedure we obtain retracts  $A_m$  of  $\mathcal{M}([0, 1]^2)$  whose maximal ideal space is a closed domain in  $[0, 1]^2$ , and whose multiplicity and index are finite and arbitrarily large.  $\square$

**Corollary 5.2.** *Adopt the notation of (16)-(17). For each  $n = 1, 2, \dots$  let the triangle  $U_n \subseteq [0, 1]^2$  be defined by  $U_n = \text{conv}([0, 0, 1], [1, 0, F_n], [n, 1, F_{n+1}])$ . Then  $2^n \leq \iota(\mathcal{M}(U_n)) = \iota(\mathcal{M}_{\mathbb{R}}(U_n)) \in \mathbb{Z}$ .*

*Proof.* By [24, Lemma 3.6], the retract  $A_n = \text{range}(- \circ \sigma^{(n)})$  of  $\mathcal{M}([0, 1]^2)$  in the foregoing theorem is isomorphic to  $\mathcal{M}(U_n)$ . So  $\iota(\mathcal{M}(U_n)) \geq 2^n$ . The preservation properties of the  $\Gamma$  functor ensure that  $\iota(\mathcal{M}_{\mathbb{R}}(U_n)) = \iota(\mathcal{M}(U_n))$ .  $\square$

**Corollary 5.3.** *For every  $j = 1, 2, \dots$ , there is retract  $R_j$  of the free unital  $\ell$ -group  $\mathcal{M}_{\mathbb{R}}([0, 1]^2)$  such that  $\iota(R_j) > j$ , and the maximal spectral space of  $R_j$  is a closed domain in  $[0, 1]^2$ .*

While the index of a finitely generated projective MV-algebra arises by taking the *sup* of multiplicities, taking the *inf* is of little interest:

**Proposition 5.4.** *Let  $A$  be a retract of  $\mathcal{M}([0, 1]^n)$ , say  $A = \text{range}(- \circ \rho)$  for some  $\mathbb{Z}$ -retraction  $\rho$  of  $[0, 1]^n$  onto the rational polyhedron  $P$ . Suppose  $P$  is a closed domain in  $[0, 1]^n$ . Then  $A$  has an isomorphic copy  $A' = \text{range}(- \circ \rho')$  where  $\rho'$  is a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  onto  $P$  and the multiplicity of  $A'$  is equal to 1.*

*Proof.* If the multiplicity of  $A$  is 1 we are done. Otherwise, let  $m > 1$  be the multiplicity of  $A$ . The finiteness of  $m$  follows from Theorem 2.3 since by assumption,  $P = \text{cl}(\text{int}(P))$ . By Theorem 2.1, there are exactly  $m$  rational polyhedra  $P = Q_1, Q_2, \dots, Q_m \subseteq [0, 1]^n$  such that for each  $i = 1, \dots, m$ ,  $\rho \upharpoonright Q_i$  is a  $\mathbb{Z}$ -homeomorphism of  $Q_i$  onto  $P$ . Now consider  $Q_2$ . By the final part of the proof of Theorem 2.3 some connected component, say  $Q$ , of the interior of  $Q_2$  is disjoint from the interior of  $P$ . Let  $\nabla$  be a regular triangulation of  $[0, 1]^n$  having the following properties:

- (i)  $\nabla$  linearizes  $\rho$  (i.e.,  $\rho$  is linear over each simplex of  $\nabla$ );
- (ii)  $\nabla$  has an  $n$ -simplex  $T = \text{conv}(t_0, t_1, \dots, t_n)$  lying in the interior of  $Q$ , where  $d = \text{den}(t_0)$ ;
- (iii)  $\nabla$  also has a vertex  $t^* \in Q \setminus T$  of denominator  $d$ .

The existence of  $\nabla$  is ensured by [24, Proposition 3.2]. Applying [24, Lemma 3.7] to the regular simplex  $T$  and to the  $n + 1$  rational points  $t^*, t_1, \dots, t_n$  we obtain a  $\mathbb{Z}$ -map  $\zeta: [0, 1]^n \rightarrow [0, 1]^n$  such that  $\zeta$  is identity over  $[0, 1]^n \setminus \text{int}(T)$ ,  $\zeta(t_0) = t^*$ , and  $\zeta(T)$  is contained in  $Q$ . It follows that  $\zeta \upharpoonright U$  is not one-one.

Thus the composite  $\mathbb{Z}$ -map  $\rho^{(1)} = \rho \circ \zeta$ , while being a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  onto  $P$ , does not act  $\mathbb{Z}$ -homeomorphically over  $Q_2$ . By Theorem 2.1, the multiplicity of the retract  $A_1 = \text{range}(- \circ \rho^{(1)})$  is equal to  $m - 1$ . Proceeding in this way we can find a  $\mathbb{Z}$ -retraction  $\rho^{(m-1)}$  of  $[0, 1]^n$  onto  $P$  such that the multiplicity of the retract  $A_{m-1} = \text{range}(- \circ \rho^{(m-1)})$  is equal to 1. Since all  $\mathbb{Z}$ -retractions  $\rho^{(1)}, \dots, \rho^{(m-1)}$  are onto the same rational polyhedron  $P$ ,  $A_{m-1}$  is isomorphic to  $A$ . Now set  $A' = A_{m-1}$ .  $\square$

## 6. COMPARING RETRACTS OF FREE MV-ALGEBRAS AND UNITAL $\ell$ -GROUPS

Two sets  $A, B$  are said to be *comparable* if either  $A \subseteq B$  or  $B \subseteq A$ .

**Proposition 6.1.** *Any two  $\mathbb{Z}$ -homeomorphic comparable rational polyhedra  $P, Q \subseteq [0, 1]^n$  are equal. However, two isomorphic comparable finitely presented subalgebras of  $\mathcal{M}([0, 1]^n)$  need not be equal.*

*Proof.* Suppose  $P \subseteq Q$  and  $P \neq Q$ . Then for some suitably large integer  $d$  the number of points of denominator  $d$  in  $P$  is strictly less than in  $Q$ . By [24, Proposition 3.15],  $P$  and  $Q$  are not  $\mathbb{Z}$ -homeomorphic. For the second statement, the subalgebra of  $\mathcal{M}([0, 1])$  generated by  $x \oplus x$  is isomorphic to  $\mathcal{M}([0, 1])$  but is not equal to  $\mathcal{M}([0, 1])$ .  $\square$

**Theorem 6.2.** *Retracts  $A, B$  of  $\mathcal{M}([0, 1]^n)$  are equal iff they are comparable and isomorphic.*

*Proof.* For the nontrivial direction, assume  $A \cong B$  and  $A \subseteq B$ , with the intent of proving  $A = B$ . For suitable McNaughton functions  $\sigma_1, \dots, \sigma_n$  and  $\tau_1, \dots, \tau_n$  with  $\sigma \circ \sigma = \sigma$  and  $\tau \circ \tau = \tau$  we can write  $A = \text{gen}(\sigma_1, \dots, \sigma_n)$  and  $B = \text{gen}(\tau_1, \dots, \tau_n)$ . The restriction to  $B$  of the retraction  $- \circ \sigma: \mathcal{M}([0, 1]^n) \rightarrow A$  is a retraction of  $B$  onto  $A$ , and we have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{(- \circ \sigma) \upharpoonright B} & A \\ \text{id} \uparrow & & \uparrow \text{id} \\ B & \xleftarrow{\text{inclusion}} & A \end{array}$$

Dually, [21], we get the commutative diagram

$$\begin{array}{ccc} R_\tau & \xleftarrow{\epsilon} & R_\sigma \\ \text{id} \downarrow & & \downarrow \text{id} \\ R_\tau & \xrightarrow{\delta} & R_\sigma \end{array} \tag{19}$$

Since  $A \subseteq B$ , from [6, Theorem 3.2(ii)] it follows that  $\delta$  is onto  $R_\sigma$ . By [6, Theorem 3.5],  $\epsilon$  is one-one and preserves denominators. Since  $A \cong B$ , by the (Cantor-Bernstein) theorem [6, Theorem 3.7],  $\epsilon$  is a  $\mathbb{Z}$ -homeomorphism of  $R_\sigma$  onto  $R_\tau$ ,

$$\epsilon: R_\sigma \cong_{\mathbb{Z}} R_\tau.$$

Now from (19) it follows that  $\delta$  and  $\epsilon$  are inverses of each other, whence

$$\delta: R_\tau \cong_{\mathbb{Z}} R_\sigma.$$

Therefore, the inclusion map of  $A$  into  $B$  (which is the dual of  $\delta$ ) is surjective, and  $A = B$ .  $\square$

The following result is a special case of [9, Theorem 4.6]. We have included it here because of its simple proof.

**Proposition 6.3.** *Let  $A$  be a separating retract of  $\mathcal{M}([0, 1]^n)$ , in the sense that for all distinct  $x, y \in [0, 1]^n$  there is  $f \in A$  with  $f(x) \neq f(y)$ . Then  $A$  coincides with  $\mathcal{M}([0, 1]^n)$ .*

*Proof.* By hypothesis we have a retraction  $\epsilon$  of  $\mathcal{M}([0, 1]^n)$  onto  $A$ . Letting  $\sigma_i = \epsilon(\pi_i)$ , ( $i = 1, \dots, n$ ), and recalling the notational stipulations in the introductory part of Section 2, the retraction  $\epsilon$  determines the  $\mathbb{Z}$ -retraction  $\mathcal{Z}_\epsilon = \sigma = (\sigma_1, \dots, \sigma_n): [0, 1]^n \rightarrow [0, 1]^n$ , and we can write  $A = \text{gen}(\sigma_1, \dots, \sigma_n)$ . If  $R_\sigma = [0, 1]^n$  we are done, and  $\sigma$  is identity on  $[0, 1]^n$ . If  $R_\sigma$  is strictly contained in  $[0, 1]^n$  then some rational point  $r \in [0, 1]^n$  of sufficiently high denominator does not belong to  $R_\sigma$ . Since  $\sigma(r)$  lies in  $R_\sigma$ , necessarily  $\sigma(r) \neq r$ . Since  $\sigma(r) = \sigma(\sigma(r))$  then for each  $f \in A$  we must have  $f(r) = f(\sigma(r))$ , because  $f$  has the form  $g \circ \sigma$  for some  $g \in \mathcal{M}([0, 1]^n)$ . We conclude that  $A$  is not a separating subalgebra of  $\mathcal{M}([0, 1]^n)$ .  $\square$

**Proposition 6.4.** *For any two  $\mathbb{Z}$ -retractions  $\sigma \neq \tau$  of  $[0, 1]^n$  with equal range, the retracts  $A_\sigma = \text{gen}(\sigma_1, \dots, \sigma_n)$  and  $A_\tau = \text{gen}(\tau_1, \dots, \tau_n)$  are (isomorphic and) incomparable.*

*Proof.* Isomorphism immediately follows from [24, Corollary 3.10]. Concerning incomparability, the solutions of the equation

$$\xi \circ \sigma = \sigma$$

in the unknown  $\xi = (\xi_1, \dots, \xi_n): [0, 1]^n \rightarrow [0, 1]^n$ ,  $\xi_i \in \mathcal{M}([0, 1]^n)$ , are precisely those elements of  $(\mathcal{M}([0, 1]^n))^n$  which act identically on the range  $R_\sigma$ . If  $A_\tau$  is a subalgebra of  $A_\sigma$  then for some  $\chi = (\chi_1, \dots, \chi_n)$  with  $\chi_i \in \mathcal{M}([0, 1]^n)$  we have  $\chi \circ \sigma = \tau$ , because  $\{\sigma_1, \dots, \sigma_n\}$  is a generating set of  $A_\sigma$ . Over the polyhedron  $R_\sigma = R_\tau$ , the function  $\chi$  must act identically, because so do  $\sigma$  and  $\tau$ . Similarly, for each  $y \in [0, 1] \setminus R_\sigma$  the point  $\sigma(y)$  lies in  $R_\sigma$ . We have proved the identity  $(\chi \circ \sigma)(y) = \sigma(y)$  for all  $y \in [0, 1]^n$ . From  $\chi \circ \sigma = \sigma$  and  $\chi \circ \sigma = \tau$  we get  $\sigma = \tau$ .  $\square$

For any fixed  $\mathbb{Z}$ -retract  $P \subseteq [0, 1]^n$  the set  $\Omega_P$  of MV-algebras is defined by

$$\Omega_P = \{\text{gen}(\sigma_1, \dots, \sigma_n) \mid \sigma \text{ any possible } \mathbb{Z}\text{-retraction of } [0, 1]^n \text{ onto } P\}.$$

By duality, any two algebras in  $\Omega_P$  are isomorphic.

**Proposition 6.5.** *In general, the intersection of two MV-algebras in  $\Omega_P$  need not be in  $\Omega_P$ . The smallest MV-algebra containing two MV-algebras in  $\Omega_P$  need not be in  $\Omega_P$ .*

*Proof.* For both statements we have examples already for  $n = 1$ .

For the first statement, let  $\sigma = \pi_1 \wedge \neg \pi_1$ . Then the map  $f \mapsto f \circ \sigma$  amounts to taking the mirror image of the first half of  $f$ . Let now  $\tau: [0, 1] \rightarrow [0, 1]$  act identically on the interval  $[0, 1/2]$ , then descend to 0 with slope  $-3$ , and finally vanish over  $[2/3, 1]$ . All functions  $f \in A_\sigma \cap A_\tau$  are symmetric around the axis  $y = 1/2$ , and are constant over the interval  $[2/3, 1]$ , so they are also constant over the interval  $[0, 1/3]$ . As a consequence,  $A_\sigma \cap A_\tau$  does not have a maximal quotient isomorphic to  $\Gamma(\mathbb{Z}_{\frac{1}{3}}, 1)$ . By [24, Lemma 3.6], every MV-algebra  $A$  in  $\Omega_{[0, 1/2]}$  is isomorphic to  $\mathcal{M}([0, 1/2])$ . So, in particular,  $A$  has a maximal quotient isomorphic to  $\Gamma(\mathbb{Z}_{\frac{1}{3}}, 1)$ . So  $A_\sigma \cap A_\tau \notin \Omega_{[0, 1/2]}$ .

For the second statement take two different  $\mathbb{Z}$ -retractions  $\sigma, \tau$  of  $[0, 1]$  onto the same range  $[0, q] \subseteq [0, 1]$ . The interval  $[0, q]$  is a  $\mathbb{Z}$ -retract of  $[0, 1]$ . Every MV-algebra in  $\Omega_{[0, q]}$  is isomorphic to  $A_\sigma$  and hence it is projective. By duality we can write  $A_\sigma \cong A_\tau \in \Omega_{[0, q]}$ . Now for definiteness assume both  $\sigma$  and  $\tau$  to have exactly three linear pieces. We *claim* that the range  $R$  of the map  $(\sigma, \tau): [0, 1] \mapsto [0, 1]^2$  is not simply connected. As a matter of fact, let us proceed along the trajectory  $t \in [0, q] \mapsto (\sigma(t), \tau(t)) \in [0, 1]^2$  starting from the  $(0, 0)$  at time  $t = 0$ . Then we go up along the diagonal  $x_1 = x_2$  of  $[0, 1]^2$  until, at time  $t = q$ , we reach the point  $(q, q)$ ; we then go down until we reach, say, the  $x$ -axis, and finally move leftward until we reach the origin, at time  $t = 1$ . The resulting piecewise linear curve  $R = \text{range}(\sigma, \tau)$  is the perimeter of a quadrangle, whence it is not simply connected. Our claim is settled.

Let  $\text{gen}(\sigma, \tau)$  denote the subalgebra of  $\mathcal{M}([0, 1])$  generated by  $\sigma$  and  $\tau$ . This is the smallest MV-algebra containing  $A_\sigma \cup A_\tau$ . By [24, Lemma 3.6] we have the isomorphism  $\text{gen}(\sigma, \tau) \cong \mathcal{M}(R)$ . By [24, Corollary 4.18], the maximal spectrum of  $\text{gen}(\sigma, \tau)$  is homeomorphic to  $R$ , so it is not simply connected, and  $\text{gen}(\sigma, \tau)$  is not projective. We conclude that  $\text{gen}(\sigma, \tau) \notin \Omega_{[0, q]}$ .  $\square$

## 7. DECISION PROBLEMS FOR PROJECTIVE ALGEBRAS

Unless otherwise specified, all MV-terms in this section are in the same variables  $X_1, \dots, X_n$ . We use the adjective “decidable” (resp., “computable”) as an abbreviation of “Turing decidable” (resp., “Turing computable”).

**Proposition 7.1.** *The following problem is decidable:*

INSTANCE : MV-terms  $t_1, \dots, t_n$ .

QUESTION : Is the map  $\hat{t} = (\hat{t}_1, \dots, \hat{t}_n)$  a  $\mathbb{Z}$ -retraction of  $[0, 1]^n$ ?

*Proof.* Checking the idempotency property  $\hat{t} \circ \hat{t} = \hat{t}$  amounts to deciding whether the MV-term  $t_i \leftrightarrow t_i \circ \hat{t}$  is a tautology in infinite-valued Łukasiewicz logic ( $i = 1, \dots, n$ ). The latter problem is decidable, [12, Corollary 4.5.3].  $\square$

The foregoing innocent looking result should be contrasted with the following:

**Proposition 7.2.** *When a rational polyhedron  $R \subseteq [0, 1]^n$  is presented as a union of rational simplexes in  $[0, 1]^n$ , or even by a rational triangulation, checking whether  $R$  is a  $\mathbb{Z}$ -retract is not a decidable problem.*

*Proof.* As is well known,  $R$  is contractible iff it is a retract of  $[0, 1]^n$ , [8, Proposition 5.1]. Using both directions of the characterization theorems of  $\mathbb{Z}$ -retracts, (respectively in [8] and [7]) it follows that  $R$  is a  $\mathbb{Z}$ -retract iff it is contractible and satisfies the following two conditions:

- (i)  $R$  has a nonempty intersection with the set of vertices of  $[0, 1]^n$ ;
- (ii)  $R$  has a *strongly regular* triangulation i.e., [8, Definition 4.1] a regular triangulation  $\Delta$  such that the greatest common divisor of the vertices of each maximal simplex of  $\Delta$  is equal to 1.

Property (i) is trivially decidable. Also (ii) is decidable, because it is equivalent to the strong regularity of every regular triangulation of  $R$ .

By way of contradiction, assume the  $\mathbb{Z}$ -retract problem is decidable. Then we can decide the contractibility of rational polyhedra in  $[0, 1]^n$ , whence the contractibility of rational polyhedra in  $\mathbb{R}^n$  would be a decidable problem. This contradicts [28, p.242].  $\square$

**Proposition 7.3.** *The following problem is decidable:*

INSTANCE : MV-terms  $t_1, \dots, t_n$  such that the map  $\hat{t}: [0, 1]^n \rightarrow [0, 1]^n$  is idempotent (a decidable hypothesis, by Proposition 7.1). Let  $\mu_A$  denote the maximal spectrum of the MV-algebra  $A \subseteq \mathcal{M}([0, 1]^n)$  generated by  $\hat{t}_1, \dots, \hat{t}_n$ .

QUESTION : Is  $\mu_A$  homeomorphic to a closed domain in  $[0, 1]^n$ ?

*Proof.* By [24, Corollary 4.18] there is a homeomorphism of  $\mu_A$  onto the set  $E = \{x \in [0, 1]^n \mid x = \hat{t}(x)\} = R_{\hat{t}}$ . The rational polyhedron  $E$  can be computed from the input MV-terms  $t_1, \dots, t_n$ . By [24, Lemma 18.1], a (regular) triangulation  $\nabla$  of  $E$  can be computed. Then  $\mu_A \cong E$  is a closed domain iff all maximal simplexes of  $\nabla$  are  $n$ -dimensional. This property is decidable.  $\square$

**Theorem 7.4.** *Let  $\sigma = (\hat{s}_1, \dots, \hat{s}_n)$  be the  $\mathbb{Z}$ -retraction of  $[0, 1]^n$  determined by the MV-terms  $s_1, \dots, s_n$ . Let  $P = R_\sigma$  be the range of  $\sigma$ , and  $A = \text{gen}(\hat{s}_1, \dots, \hat{s}_n)$  be the retract of  $\mathcal{M}([0, 1]^n)$  associated to  $\sigma$ . If  $P$  is a closed domain, the multiplicity  $r(A)$  is computable from the input terms  $s_1, \dots, s_n$ .*

*Proof.* Given the input terms  $s_1, \dots, s_n$  the idempotency of  $\sigma$  is decidable by Proposition 7.1, and so is the hypothesis that  $P$  is a closed domain, by Proposition 7.3. Let us write  $P = R_\sigma$ . For any  $\mathbb{Z}$ -retraction  $\tau = (\tau_1, \dots, \tau_n)$  of  $[0, 1]^n$  we have  $\text{gen}(\tau_1, \dots, \tau_n) = \text{gen}(\hat{s}_1, \dots, \hat{s}_n)$  iff  $\sigma \upharpoonright R_\tau$  is a  $\mathbb{Z}$ -homeomorphism of  $R_\tau$  onto  $R_\sigma$ . This is proved in Theorem 2.1. Let  $\Delta$  be a regular triangulation of  $[0, 1]^n$  that

linearizes  $\sigma$ , (i.e.,  $\sigma$  is linear over each simplex of  $\Delta$ .) By [24, Lemma 18.1],  $\Delta$  is computable from the input MV-terms  $s_i$ . Let the subcomplex  $\nabla \subseteq \Delta$  of simplexes be defined by

$$\nabla = \{S \in \Delta \mid \sigma \upharpoonright S \text{ is a } \mathbb{Z}\text{-homeomorphism of } S \text{ onto } \sigma(S)\}.$$

Also  $\nabla$  is computable from the input MV-terms  $s_i$ . Let  $|\nabla|$  denote the support of  $\nabla$ ,

$$|\nabla| = \bigcup \{S \mid S \in \nabla\}.$$

*Claim 1.* Suppose the rational polyhedron  $Q \subseteq [0, 1]^n$  satisfies  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} P$ . Then  $Q \subseteq |\nabla|$ .

As a matter of fact, since  $P$  is a closed domain in  $[0, 1]^n$  then so is  $Q$ . Fix  $x \in \text{int}(Q)$  together with an  $n$ -simplex  $S \in \Delta$  such that  $x \in S$ . There is a rational simplex  $T$  satisfying  $T \subseteq Q \cap S$ . From  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} P$  we get  $\sigma \upharpoonright T: T \cong_{\mathbb{Z}} \sigma(T)$ . Since  $T$  and  $S$  are  $n$ -simplexes and  $\sigma$  is linear over  $S$  (because  $S \in \Delta$  and  $\Delta$  linearizes  $\sigma$ ) then  $\sigma \upharpoonright S: S \cong_{\mathbb{Z}} \sigma(S)$ . We have thus shown that  $\text{int}(Q)$  is contained in  $|\nabla|$ . Since  $Q$  is a closed domain and  $|\nabla|$  is closed then  $Q$  is contained in  $|\nabla|$ , and our claim is settled.

We now strengthen Claim 1 as follows:

*Claim 2.* Suppose the rational polyhedron  $Q \subseteq [0, 1]^n$  satisfies  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} P$ . Then  $Q = \bigcup \{S \in \nabla \mid S \subseteq Q\}$ .

To prove this claim, again fix  $x \in \text{int}(Q)$ . By Claim 1 there is  $S \in \nabla$  with  $x \in S$ . By way of contradiction suppose  $S$  is not contained in  $Q$ . Then by Claim 2 in Theorem 2.3 (using the connectedness of  $\text{int}(S)$ ) there is  $y \in \text{int}(S)$  satisfying  $y \in Q \setminus \text{int}(Q)$ . From  $\sigma \upharpoonright Q: Q \cong_{\mathbb{Z}} P$  it follows that  $\sigma(y) \in P \setminus \text{int}(P)$ . From  $\sigma \upharpoonright S: S \cong_{\mathbb{Z}} \sigma(S)$  it follows that  $\sigma(y) \in \text{int}(\sigma(S)) \subseteq \text{int}(P)$ . This contradiction settles Claim 2.

To conclude the proof, for each subset  $\mathcal{S}$  of  $\nabla$  only consisting of  $n$ -dimensional simplexes, it is decidable whether  $\sigma \upharpoonright \bigcup \mathcal{S}$  is a  $\mathbb{Z}$ -homeomorphism of  $\bigcup \mathcal{S}$  onto  $P$ . Injectivity is equivalent to the following property: For any two distinct  $k$  simplexes  $V, W \in \mathcal{S}$ , from  $\text{relint}(V) \cap \text{relint}(W) = \emptyset$  it must follow that  $\sigma(\text{relint}(V)) \cap \sigma(\text{relint}(W)) = \emptyset$ . This amounts to a routine linear algebra problem involving intersections of rational hyperplanes in  $\mathbb{R}^n$ , once  $V$  and  $W$  are presented as intersections of rational hyperplanes—in an effective way as in [24, Lemma 18.1]. Once the injectivity of  $\sigma \upharpoonright \bigcup \mathcal{S}$  has been verified, we check surjectivity by computing the  $n$ -dimensional Lebesgue measure  $\lambda$  of the union of all  $n$ -dimensional simplexes in  $\mathcal{S}$ . This is computable because  $\Delta$  is a rational (actually, a regular) triangulation. We finally check that  $\lambda$  is equal to the measure of  $\bigcup \{\sigma(T) \mid T \in \mathcal{S}\}$ . This, too, is computable, once the set  $\bigcup \{\sigma(T) \mid T \in \mathcal{S}\}$  has been equipped with a regular triangulation. In this way, some Turing machine can compute the set  $\Lambda = \mathcal{S}_1, \dots, \mathcal{S}_w$  of all subsets  $\mathcal{S}$  of  $\nabla$  such that  $\sigma \upharpoonright \bigcup \mathcal{S}$  is a  $\mathbb{Z}$ -homeomorphism of  $\bigcup \mathcal{S}$  onto  $P$ . By Theorem 2.1, the number of elements in  $\Lambda$  coincides with the multiplicity of  $A$ ,  $w = r(A)$ .  $\square$

**Proposition 7.5.** *The following problem is decidable:*

**INSTANCE :** MV-terms  $t_1, \dots, t_n$  such that the map  $\hat{t}: [0, 1]^n \rightarrow [0, 1]^n$  is idempotent, and the maximal spectral space  $\mu_A$  of  $A = \text{gen}(\hat{t}_1, \dots, \hat{t}_n)$  is homeomorphic to a closed domain (both conditions being decidable, respectively by Proposition 7.1 and 7.3).

**QUESTION :** Let  $\text{int}(\mu_A)$  denote the interior of  $\mu_A$ . Is  $\text{int}(\mu_A)$  connected?

*Proof.* Again replace  $\mu_A$  by its homeomorphic copy given by the rational polyhedron  $E = \{x \in [0, 1]^n \mid x = \hat{t}(x)\} = R_{\hat{t}}$ . Compute a rational triangulation  $\Delta$  of  $E$ . Verify the closed domain hypothesis by checking that all maximal simplexes of  $\Delta$  are  $n$ -dimensional. Call  $\Delta^{(n)}$  the collection of all these  $n$ -simplexes, ordered lexicographically. Inductively, letting  $X_k$  be the set of the first  $k$  simplexes of  $\Delta^{(n)}$ , add to  $X_k$  the first simplex of  $\Delta^{(n)}$  which shares a facet with some simplex of  $X_k$ . Denote by  $X_{k+1}$  the new set of  $n$ -simplexes thus obtained. Note that  $X_{k+1}$  has a connected interior if so does  $X_k$ . After  $u$  steps no more  $n$ -simplexes can be added to  $X_u$ . Then check that  $X_u$  equals  $\Delta^{(n)}$ .  $\square$

**Proposition 7.6.** *The following problem is decidable:*

INSTANCE : *MV-terms  $s_1, \dots, s_n$  and  $t_1, \dots, t_n$  providing  $\mathbb{Z}$ -retractions  $\hat{s}, \hat{t}$  of  $[0, 1]^n$  (a decidable hypothesis, by Proposition 7.1).*

QUESTION : *Do these two  $\mathbb{Z}$ -retractions have the same range?*

*Proof.* The ranges of  $\hat{s}$  and  $\hat{t}$  are computable from the input terms  $s_1, \dots, s_n$  and  $t_1, \dots, t_n$ . It is decidable whether the two rational polyhedra  $R_{\hat{s}}$  and  $R_{\hat{t}}$  coincide, [24, Corollary 18.4].  $\square$

**Proposition 7.7.** *The following problem is decidable:*

INSTANCE : *MV-terms  $s_1, \dots, s_n$ , and  $t_1, \dots, t_n$  yielding  $\mathbb{Z}$ -retractions  $\hat{s}$  and  $\hat{t}$  of  $[0, 1]^n$  with the same range, (both assumption being decidable, by Propositions 7.1 and 7.6).*

QUESTION : *Does the MV-algebra generated by  $\hat{s}_1, \dots, \hat{s}_n$  coincide with the MV-algebra generated by  $\hat{t}_1, \dots, \hat{t}_n$ ?*

*Proof.* By Proposition 6.4, the answer is positive answer iff  $\hat{s} = \hat{t}$ . This in turn is equivalent to checking whether the MV-term  $s_i \leftrightarrow t_i$  is a tautology for all  $i = 1, \dots, n$ , which is a decidable problem, [12, Corollary 4.5.3].  $\square$

Dropping the hypothesis that  $\hat{s}$  and  $\hat{t}$  have the same range, the problem remains decidable, yet with a much subtler proof:

**Theorem 7.8.** *The following problem is decidable:*

INSTANCE : *MV-terms  $s_1, \dots, s_n$ , and  $t_1, \dots, t_n$  determining  $\mathbb{Z}$ -retractions of  $[0, 1]^n$   $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n)$  and  $\hat{t} = (\hat{t}_1, \dots, \hat{t}_n)$ , (a decidable condition, by Proposition 7.1).*

QUESTION : *Does the MV-algebra  $A$  generated by  $\hat{s}_1, \dots, \hat{s}_n$  coincide with the MV-algebra  $B$  generated by  $\hat{t}_1, \dots, \hat{t}_n$ ?*

*Proof.* Let  $P = R_{\hat{s}}$  be the range of  $\hat{s}$  and  $Q = R_{\hat{t}}$  be the range of  $\hat{t}$ . If  $P$  coincides with  $Q$  (a decidable condition, by Proposition 7.6) then Proposition 7.7 shows that the problem  $A = B$  is decidable. So it is sufficient to argue in case  $P \neq Q$ . We have

$$A = B \text{ iff } \hat{s} \upharpoonright Q \text{ is a } \mathbb{Z}\text{-homeomorphism of } Q \text{ onto } P, \text{ and } \hat{t} = (\hat{s} \upharpoonright Q)^{-1} \circ \hat{s}. \quad (20)$$

The  $(\Rightarrow)$ -direction is proved in Theorem 2.1. For the  $(\Leftarrow)$ -direction, the hypothesis shows that  $(\hat{s} \upharpoonright Q) \circ \hat{t} = \hat{s}$ , whence  $A = \text{gen}(\hat{s}_1, \dots, \hat{s}_n) = \text{gen}(\rho_1, \dots, \rho_n) = B$ .

Next, in order to check the right-hand side of (20) we proceed as follows:

- (i) Using the effective procedure of [24, Corollary 2.9], we compute a regular triangulation  $\Lambda$  of  $Q$  such that  $\hat{s} \upharpoonright Q$  is linear over each simplex of  $\Lambda$ . In the light of the characterization of  $\mathbb{Z}$ -homeomorphisms, [24, Proposition 3.15], we then check whether
  - each maximal simplex  $M$  of  $\Lambda$  is sent by  $\hat{s}$  onto a regular simplex  $\Lambda(M) \subseteq P$  with preservation of the denominators of the vertices of  $M$ ;

- the relative interiors of any two distinct simplexes  $M', M''$  of  $\Lambda$  are sent to disjoint simplexes  $\Lambda(M'), \Lambda(M'')$ ;
  - the  $i$ -dimensional rational measure [25] of  $\Lambda(Q)$  coincides with the  $i$ -dimensional rational measure of  $Q$ , for each  $i = 0, 1, \dots, n$ .
- (ii) The three conditions above are necessary and sufficient for  $\hat{s} \upharpoonright Q$  to be a  $\mathbb{Z}$ -homeomorphism of  $Q$  onto  $P$ .
- (iii) Using the extension argument, [24, Theorem 5.8(ii)] it is easy to compute MV-terms  $r_1, \dots, r_n$  such that the  $\mathbb{Z}$ -map  $\hat{r} = (\hat{r}_1, \dots, \hat{r}_n)$  coincides with  $(\hat{s} \upharpoonright Q)^{-1}$  over  $P$ .
- (iv) The verification of the identity  $\hat{t} = (\hat{s} \upharpoonright Q)^{-1} \circ \hat{s}$  now amounts to checking whether the MV-term  $t_i \leftrightarrow r_i \circ (s_1, \dots, s_n)$  is a tautology in Łukasiewicz logic for all  $i = 1, \dots, n$ , which, as we have seen, is decidable.

The proof is complete.  $\square$

Replacing identity by isomorphism in the foregoing theorem we have an open problem:

**Problem 7.9.** The following problem is open:

**INSTANCE :** MV-terms  $s_1, \dots, s_n$ , and  $t_1, \dots, t_n$  yielding  $\mathbb{Z}$ -retractions  $\hat{s}$  and  $\hat{t}$  of  $[0, 1]^n$ , (a decidable condition, by Proposition 7.1).

**QUESTION :** Is the subalgebra of  $\mathcal{M}([0, 1]^n)$  generated by  $\hat{s}_1, \dots, \hat{s}_n$  isomorphic to the subalgebra of  $\mathcal{M}([0, 1]^n)$  generated by  $\hat{t}_1, \dots, \hat{t}_n$ ?

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